Primitive Orthogonal Idempotents for R-Trivial Monoids

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History of the problem

Goal: Construct primitive orthogonal idempotents of the Hecke algebra $H_W(0)$.

Norton 1979: Constructs analogue of Young Idempotents η_α for $H_n(0)$ (type A). $H_n(0)\eta_\alpha$ gives all projective indecomposables (not simples), but the η_α are not idempotents nor orthogonal.

Krob-Thibon 1997: Representation theory of $H_n(0)$ is related with $QSym$ and $NSym$. (Characteristic map)

Schocker 2008: Defines WOM (Weakly Ordered Monoids) and connects *left regular bands* and *Hecke algebras at* $q = 0$ (all types).

Denton 2010: Constructs orthogonal idempotents for $H_n(0)$ ... but no relation with η_α. Extended to J-trivial monoids by Denton, Hivert, Thiéry, Schilling.

BBBS 2010: Constructs orthogonal idempotents for WOM, generalizing η_α.

Motivating Examples

Left Regular Bands

- Semigroups W such that $x^2 = x$ and $xyx = xy$ for all $x, y \in W$.
- Support map supp: $W \to L$: there is surjection onto the lattice

$$L = W/\sim \text{ where } x \sim y \text{ iff } x = xy \text{ and } y = yx$$

- Radical of $\mathbb{K}W$: $\sqrt{\mathbb{K}W} = \ker(\text{supp})$, and

$$\mathbb{K}W \Big/ \sqrt{\mathbb{K}W} \cong \mathbb{K}L.$$

- Orthogonal idempotents of $\mathbb{K}W$: first construction by Brown; later simplified by Saliola: for $J \in L$, fix x_J with $\text{supp}(x_J) = J$, and let

$$e_J := x\left(1 - \sum_{K>J} e_K\right)$$

Hecke monoids (type A)

- Generated by $T_1, T_2, \ldots, T_{n-1}$ with relations:

$$T_i^2 = T_i$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$$
$$T_iT_j = T_jT_i \quad \text{for} \quad |i-j| > 1.$$

- two maps onto the lattice of subsets of $[n-1]$:
  - Descent map: $D(T_w) = \{i : T_wT_i = T_w\}$
  - Content map: $C(T_w) = \{i : T_i \text{ occur in } T_w\}$
- Radical of $\mathbb{K}W$: $\sqrt{\mathbb{K}W} = \ker(C)$, and

$$\mathbb{K}W \Big/ \sqrt{\mathbb{K}W} \cong \mathbb{K}L.$$

Weakly Ordered Monoids (WOM) [Schocker]

Definition of WOM

Manfred Schocker introduced WOM hoping to construct primitive orthogonal idempotents for Hecke algebras at $q = 0$.

Preorder: $u \leq v \iff uw = v$ for some $w \in W$

Definition: W is a **WOM** if there are a finite upper semilattice L and two maps $C, D: W \to L$ satisfying:

1. C surjective morphism of monoids.
2. $uv \leq u$ and $u \leq uv \implies C(v) \leq D(u)$.
3. $C(v) \leq D(u) \implies uv = u$.

Examples and Properties

Examples:

- Left regular bands: $C = D = \text{supp}$
- Hecke monoids : C = content map ; D = descent map

Proposition A [Schocker]: If W is a WOM, then $\leq$ is an order and

$$\boxed{\sqrt{\mathbb{K}W} = \ker(C)}$$

Corollary A: $\mathbb{K}W \Big/ \sqrt{\mathbb{K}W} \cong \mathbb{K}L$ is semisimple and commutative.

WOM and R-trivial monoids

Definition. W is **R-trivial** if for all $x, y \in W$,

$$xW = yW \implies x = y$$

Proposition B: $\leq$ is an order $\iff$ W is R-trivial

Proposition C [N. M. Thiéry and B. Steinberg]:

$$\boxed{W \text{ is a WOM} \iff W \text{ is R-trivial}}$$

Analogues of Norton Elements

Let W be WOM generated by $G = \{g_1, g_2, \ldots\}$.

ω-power: If x is an element of a finite semigroup W, then there is a power x^ω of x that is idempotent:

$$x^\omega x^\omega = x^\omega$$

Number of primitive idempotents: Since $\mathbb{K}W \Big/ \sqrt{\mathbb{K}W} \cong \mathbb{K}L$ is semisimple and commutative, there is one primitive idempotent for each element of L.

Step 1. Analogues of Norton elements: for $J \in L$, define

$$\eta_J = A_J T_J$$

$$T_J := \left(\prod_{\substack{g \in G \\ C(g) \leq J}} g^\omega\right)^\omega$$

Example: For the Hecke algebra $H_7(0)$, if $J = \{1, 3, 4\}$ then

$$T_J = T_1T_3T_4T_3$$

Nice Property: $T_Jx = T_J$ for all x such that $C(x) \leq J$

Can we do that?!

$$A_J := \left(\prod_{\substack{g \in G \\ C(g) \not\leq J}} (1 - g^\omega)\right)^\omega$$

Proposition D [BBBS] A_J is well-defined.

Example: For the Hecke algebra $H_7(0)$, if $J = \{1, 3, 4\}$ then

$$A_J = \overline{T}_2\overline{T}_5\overline{T}_6\overline{T}_5 \qquad \text{where } \overline{T}_i = 1 - T_i$$

Nice Property: $xA_J = 0$ for all x such that $C(x) \not\leq J$

Constructing Idempotents

Properties of η_J:

- *Not* idempotent: A_J and T_J are both idempotents but ... η_J IS NOT.
- but *almost* orthogonal: $J \not\leq K \implies \eta_J\eta_K = 0$

Step 2. Build an idempotent:

$$P_J := \left(\sum_{n \geq 0} \eta_J(1 - \eta_J)^n\right)^2$$

Proposition E [BBBS]

$$\boxed{\eta_J^2(1 - \eta_J)^N = 0 \text{ for some } N > 0}$$

Properties of P_J:

- P_J is idempotent: $P_J^2 = P_J$ $\quad$ $\left[\text{because } \sum_{n=0}^N x(1-x)^n = 1 - (1-x)^{N+1}\right]$
- *almost* orthogonal: $J \not\leq K \implies P_JP_K = 0$.

Step 3. Orthogonalize: we apply the idempotent trick of Saliola to "orthogonalize P_J": define

$$e_J := P_J\left(1 - \sum_{K>J} e_K\right)$$

Theorem [BBBS]

$\{e_J\}_{J \in L}$ is a complete system of primitive orthogonal idempotents for $\mathbb{K}W$.

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