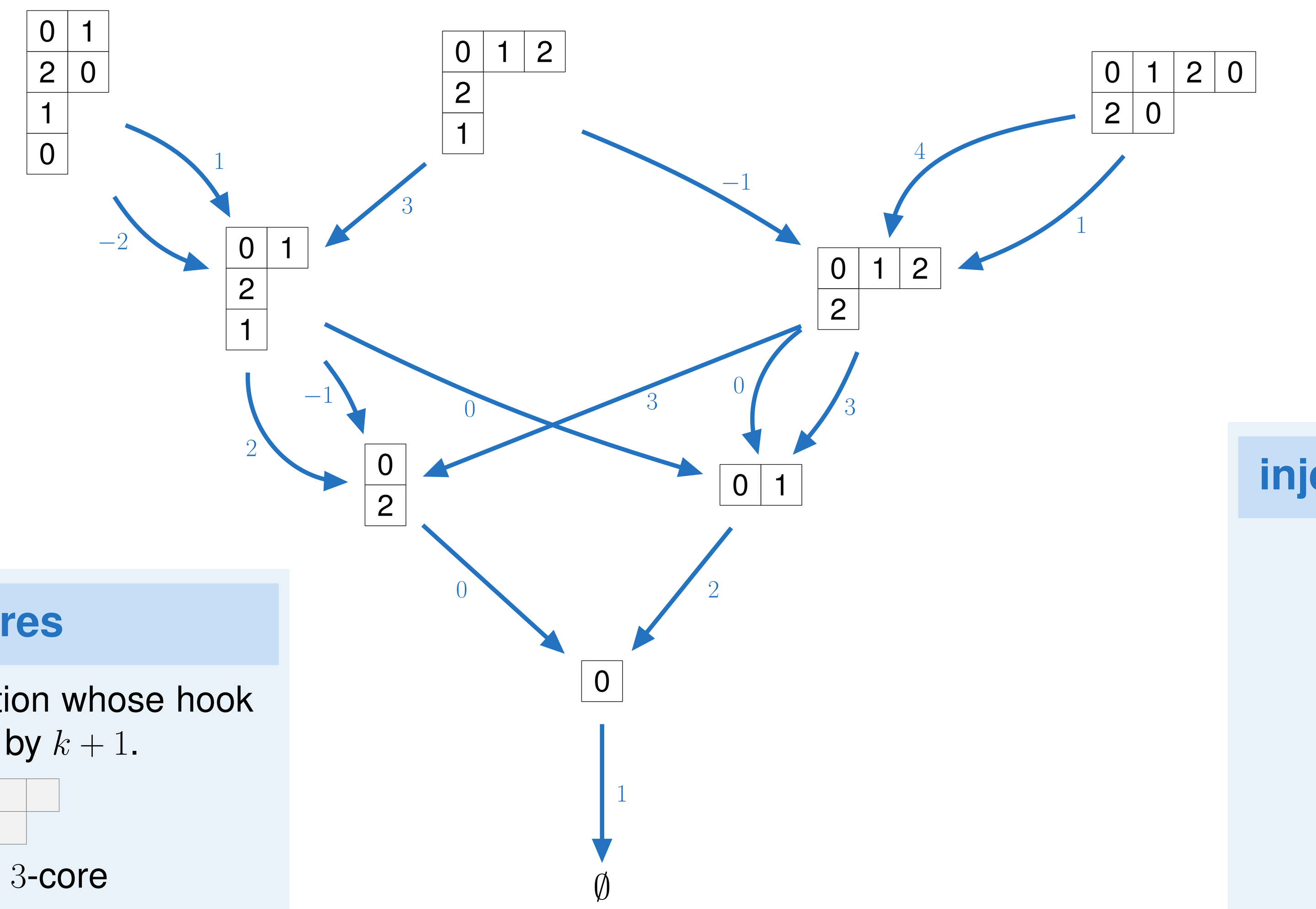
Down operators on the affine nilCoxeter algebra and expansions of noncommutative k-Schur functions

Chris Berg, Franco Saliola, Luis Serrano

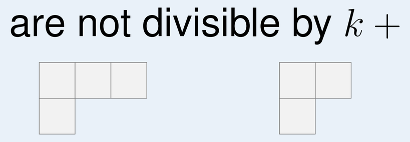
cberg@lacim.ca, saliola.franco@uqam.ca, serrano@lacim.ca

Strong $(k+1)$-cores graph $\mathfrak{C}$



$(k+1)$-cores

A $(k+1)$-*core* is a partition whose hook lengths are not divisible by $k+1$.

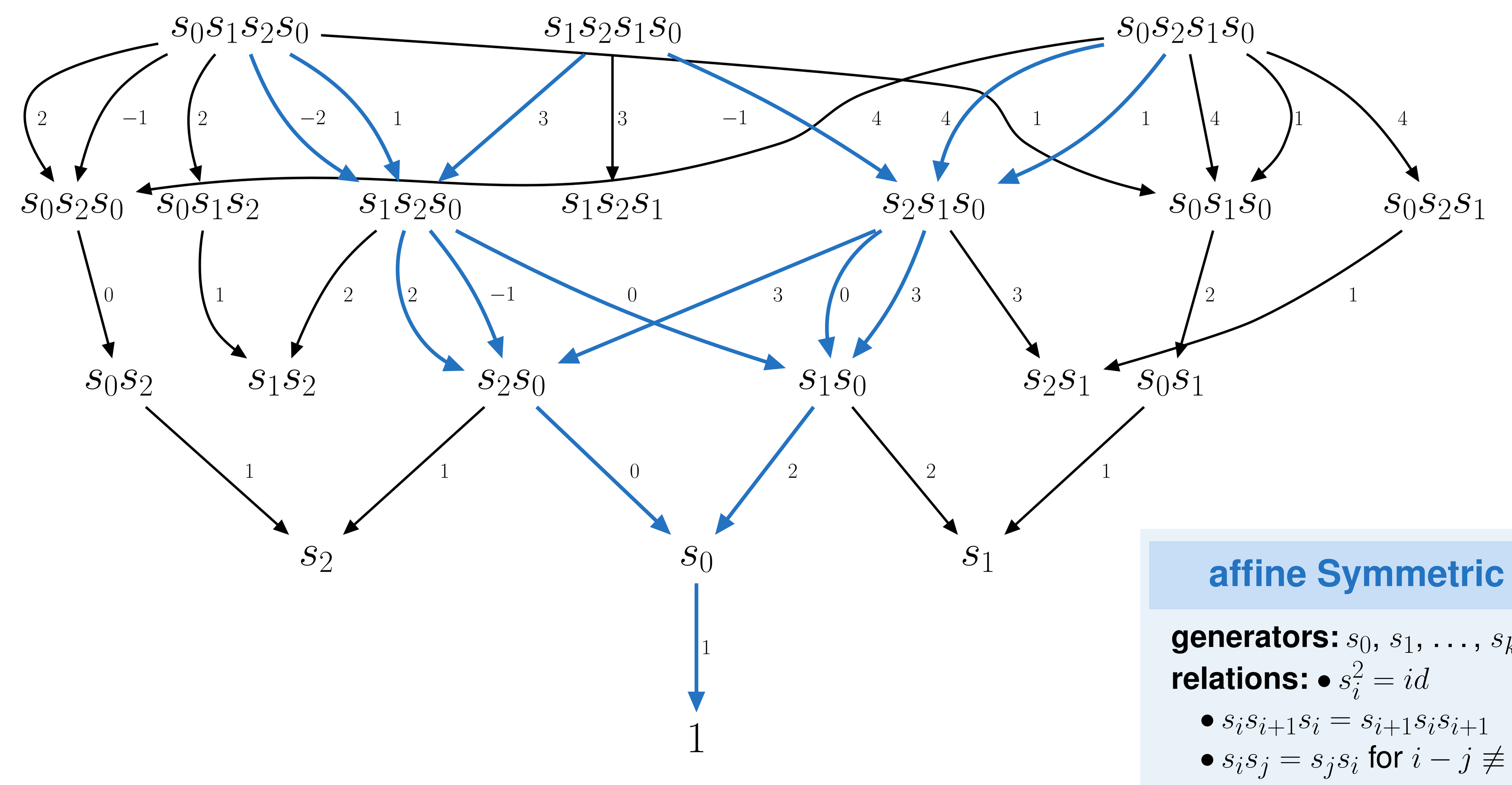


3-core   Not 3-core

injection of $(k+1)$-cores into W

0	$\longmapsto$	s_0
0 1	$\longmapsto$	$s_1 s_0$
0 1 2 / 2	$\longmapsto$	$s_2 s_1 s_0$
0 1 2 / 2 / 1	$\longmapsto$	$s_1 s_2 s_1 s_0$

Strong Bruhat Graph on the affine symmetric group W



affine Symmetric Group W

generators: $s_0, s_1, \ldots, s_k$

relations:
- $s_i^2 = id$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- $s_i s_j = s_j s_i$ for $i - j \not\equiv \pm 1 \mod k+1$

Down operators

Definition. The *ascent composition* of $v_0 \xrightarrow{\ell_1} \cdots \xrightarrow{\ell_m} v_m$ is $[i_1, i_2 - i_1, \ldots, i_j - i_{j-1}, m - i_j]$ where $i_1 < \cdots < i_j$ are the indices i_a for which $\ell_{i_a} < \ell_{i_a+1}$. For a composition J of m:

$$D_J^\Gamma(v_0) = \sum_{\text{ascents}\left(v_0 \xrightarrow{\ell_1} v_1 \xrightarrow{\ell_2} \cdots \xrightarrow{\ell_m} v_m\right) = J} v_m$$

where the sum ranges over all paths in the graph Γ starting at v_0 of length m.

Examples $(k=2)$:

$$D^W_{[1]}(s_1 s_2 s_1 s_0) = s_1 s_2 s_0 + s_1 s_2 s_1 + s_2 s_1 s_0$$
$$D^W_{[3]}(s_1 s_2 s_1 s_0) = s_2 + s_0$$
$$D^W_{[2,1]}(s_1 s_2 s_1 s_0) = s_2 + 2 s_0 + s_1$$

Property 1. D^W_J and D^W_K commute: $D^W_J \circ D^W_K = D^W_K \circ D^W_J$.

Property 2. If $uv \in W$ with u corresponding to a k-core $\mathfrak{c}_u$, then

$$D^W_J(uv) = D^{\mathfrak{C}}_J(\mathfrak{c}_u) \odot v$$

where $x \odot y = xy$ if $\text{len}(xy) = \text{len}(x) + \text{len}(y)$ and 0 otherwise.

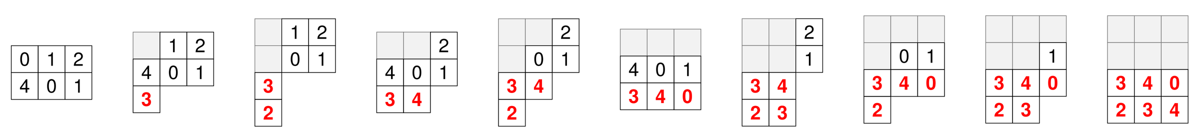
Application: Noncommutative k-Schur function expansions

Problem

The *noncommutative k-Schur functions* $\mathfrak{s}^{(k)}_\lambda$ are elements of a subalgebra of the *affine nilCoxeter algebra*. Expanding $\mathfrak{s}^{(k)}_\lambda$ into the natural basis of affine permutations is equivalent to finding the k-Littlewood–Richardson coefficients [Lam].

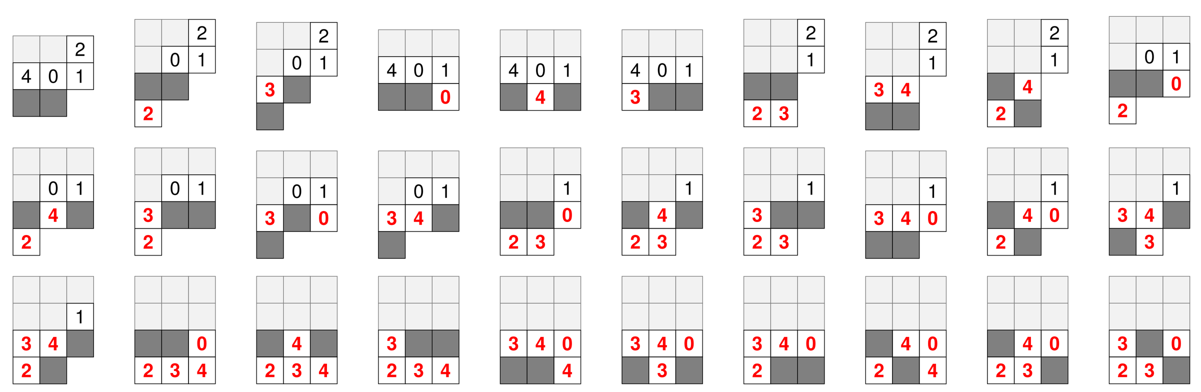
History

Berg, Bergeron, Thomas and Zabrocki proved that the expansion of $\mathfrak{s}^{(k)}_R$ for k-rectangles R is the sum of the reading words of the skew-partitions $(R \cup \lambda)/\lambda$, where $\lambda \subseteq R$. E.g., $\mathfrak{s}^{(4)}_{[3,3]}$ is the sum of the reading words of the diagrams below.

$\mathfrak{s}^{(4)}_{[3,3]} =$  $= u_1 u_0 u_4 u_2 u_1 u_0 + u_3 u_1 u_0 u_4 u_2 u_1 + \cdots$

New Result

To compute $\mathfrak{s}^{(4)}_{(3,1)}$, delete two red boxes from different columns in the diagrams; or equivalently, apply $D_{[2]}$:

$\mathfrak{s}^{(4)}_{(3,1)} = D_{[2]}\left(\mathfrak{s}^{(4)}_{[3,3]}\right) =$ 

Application: Strong Schur functions (LLMS conjectures)

Lam, Lapointe, Morse, and Shimozono generalized the k-Schur functions to a larger set of functions called the strong Schur functions $\text{Strong}_{u/v}$, where u and v of elements in W. They conjectured some of their properties which we can prove using the down operators.

Theorem 1 [Conj. 4.18 (1) LLMS] **The strong Schur functions are symmetric and monomial positive.**

$$\text{Strong}_{u/v} = \sum_\lambda \langle D^\lambda(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} m_\lambda, \text{ where } D^\lambda = D_{\lambda_1} \circ \cdots \circ D_{\lambda_l}.$$

Theorem 2 [Conj. 4.18 (2) LLMS] **The strong Schur functions lie in $\Lambda_{(k)}$.**

$$\text{Strong}_{u/v} = \sum_{\lambda \in \mathcal{B}^{(k)}} \langle \widehat{m_\lambda^\perp}(\mathbf{u}_u), \mathbf{u}_v \rangle_{\mathbb{A}} h_\lambda.$$

Theorem 3 [Conj. 4.18 (3) LLMS] Let μ, ν be k-bounded. **The strong Schur functions $\text{Strong}_{\mu/\nu}$ are k-Schur positive.** Moreover, the coefficient of $s^{(k)}_\lambda$ in $\text{Strong}_{\mu/\nu}$ is the coefficient of $\widetilde{F}_\mu$ in $\widetilde{F}_\lambda \widetilde{F}_\nu$.

$$\text{Strong}_{\mu/\nu} = \sum_{\lambda \in \mathcal{B}^{(k)}} \langle s^{(k)}_\mu, \widetilde{F}_\lambda \widetilde{F}_\nu \rangle s^{(k)}_\lambda.$$