CONSTRAINING PLANE CONFIGURATIONS IN CAD: CIRCLES, LINES, AND ANGLES IN THE PLANE*

FRANCO SALIOLA† AND WALTER WHITELEY‡

Abstract. This paper investigates the local uniqueness of designs of m-circles (lines and circles) in the plane up to inversion under a set of angles of intersection as constraints. This local behavior is studied through the Jacobian of the angle measurements in a form analogous to the rigidity matrix for a framework of points with distance constraints. After showing directly that the complete set of angle constraints on v distinct m-circles gives a matrix of rank $3v - 6$, we show that the Jacobian is column equivalent by a geometric correspondence to the rigidity matrix for a bar-and-joint framework in Euclidean 3-space. As a corollary, the complexity of the independence of angle constraints on generic plane circles is the complexity of the old unsolved combinatorial problem of generic rigidity in 3-space. This theory is not known to have a polynomial time algorithm for generic independence that offers a warning about the complexity of general systems of geometric constraints even in the plane.

Our correspondence extends to all dimensions. Angle constraints on spheres in 3-space then match the even more complex first-order theory of frameworks in 4-space. This theory is not predicted to have a polynomial time algorithm for generic points.

Key words. computer aided design, constraint, inversive geometry, circles and angles, generic rigidity, hyperbolic geometry

AMS subject classifications. Primary, 68U07; Secondary, 05C50, 51N05, 52C25

DOI. 10.1137/S0895480100374138

1. Introduction. A standard problem in computer aided design (CAD) is to find stable combinatorial techniques to decide when a set of constraints is independent, and solvable algebraically, in a reasonable time (or real time in parametric CAD programming) [16, 10]. In the study of combinatorial algorithms for independence, the classical example is points and distance in a plane or, equivalently, generic rigidity of plane frameworks. Given a graph, there is a fast combinatorial algorithm to decide the independence of the edges, as distance constraints, for almost all (generic) placements of the vertices as distinct points in the plane [15, 7, 24]. At the other extreme, there are sample problems, including points and distances in 3-space, for which there is no known polynomial time combinatorial algorithm to decide independence [7, 25, 32].

It is of interest to consider other geometric objects beyond points and other geometric constraints beyond distances (e.g., angles) and to discover the combinatorics and the geometry of their independence structure. In this paper, we consider circles of variable radii in the plane, with the angle of intersection of pairs of circles as the constraint. The case of mutually tangent circles, or even lines tangent to circles, are special cases of this problem, as outlined below. This problem of circles of variable radii with angles of intersection (including tangency) is a real problem in practical CAD programming [17, 12, 13]. We describe some new results and some important
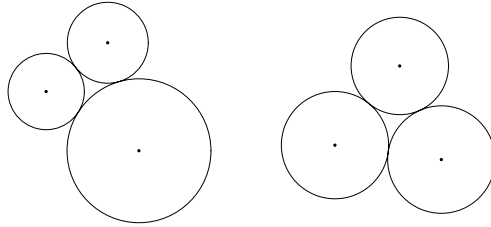

FIG. 1.1. *Are three mutually tangent circles unique up to inversion?*

connections, which we hope will offer insights for CAD programmers working on this problem, embedded in general systems of constraints.

Of course, answering this combinatorial question is only one step in the process. Once we have a combinatorial algorithm, there is still the problem of determining algebraic methods for solving the constraints for maximal independent sets of constraints and the problem of "special position" in which the constraints that are typically independent become dependent, such as the lengths of the third edge of a collinear triangle for distance constraints. We will not address either of these hard problems in detail in this paper.

Consider the two sets of mutually tangent circles in Figure 1.1. The following question arises: Is there an angle preserving map of the plane (a series of inversions) that carries the three circles on the left onto the three circles on the right preserving tangency? The question can be broadened. Suppose that the constraints on circles are not limited to tangency but include a fixed angle for each designated pair of circles. A similar question can be posed for any design: a set of circles and lines under angle constraints at their intersections.

This problem of uniqueness, which is typical of underlying problems which arise in CAD, is hard once we move to larger sets [14, 16, 17]. As an important analogy, recall that for points in the plane, constrained by distances, the general problem of "global" rigidity (uniqueness up to congruence) is unsolved, and we have no general algorithm which applies to all designs [5, 8].

A simpler problem, which is required for parametric CAD programming, is to test "local uniqueness" of the design—a property that can be predicted using the linear algebra of the Jacobian of the constraints for almost all configurations [4, 16, 23, 32]. The rank of this Jacobian matrix for the constraints, often called the rigidity matrix (for distances) or the constraint matrix (for more general constraints), is determined through two layers of analysis:

1. the combinatorial level, which determines the maximum rank by the combinatorics of which constraints are selected (in our case, a graph with vertices for the circles and an edge for each angle constraint);

2. the geometric level, which describes when the rank of the matrix drops below the maximum in (1).

Whenever the rank is maximal, the linear algebra of the Jacobian predicts whether the design is locally unique or has a continuous path of configurations satisfying the same constraints but which is geometrically inequivalent (section 4.3). Since this maximal rank occurs for "almost all" choices of the objects, this combinatorial theory gives a strong prediction of what will happen in a CAD problem for "generic" circumstances. Moreover, even if a lower rank configuration remained locally unique, the

usual numerical algorithms will become ill-determined in these nongeneric situations. In short, the rank of the Jacobian is a critical factor in any analysis of a design.

For angle constraints among circles and lines, the appropriate geometry is inversive geometry. Under inversions, the set of circles and lines goes to the set of circles and lines, and all angles are preserved. Thus, at best, angle constraints can make a design unique, or locally unique, up to inversion. Because inversive geometry may not be familiar to many readers in CAD, in section 2 we provide a brief introduction to inversions in the plane, including both synthetic and analytic representations. This is followed by an introduction to a representation of the circles as points in $\mathbb{R}^3$ (section 3).

We then present the Jacobian of the system of angle constraints, proving a basic theorem on the minimum number of constraints which makes a design unique up to local inversion: $|E| = 3|V| - 6$ (section 4).

Within this initial analysis, we find a striking similarity to the theory of rigid bar-and-joint frameworks in Euclidean 3-space with the same graph [4, 25, 30, 32]. After a brief recollection of the first-order theory of such frameworks (section 5), we present an unexpected isomorphism between the Jacobian for these three-dimensional frameworks and the constraint matrix for circles and lines with angle constraints in the plane (section 6). This permits the complete transfer of known results as well as longstanding conjectures from the first-order theory of frameworks of points and distances into this first-order theory for circles, lines, and angles (section 7). This complete transfer raises the likelihood that there is no polynomial time algorithm for determining the independence of generic patterns of angle constraints on circles in the plane, in contrast to the desirable model of a $O(|E|^2)$ algorithm offered by points and distances in the plane.

These results extend to all dimensions, giving, for example, a transfer between the first-order theory of spheres and planes in 3-space, constrained by angles, and the first-order theory of bar-and-joint frameworks in Euclidean 4-space (section 8).

To this point, the theory assumes circles with variable radii and only angle constraints. In section 9 we introduce additional constraints to fix some or all of the radii of the circles. An extension of the transfer to 3-space confirms that, with all radii fixed, the theory simplifies to the theory of the centers of the circles with distance constraints replacing the angle constraints. This special case does have fast algorithms. However, when we mix fixed and variable radii (with more than two variable circles), the theory returns to at least the complexity of spatial rigidity.

Beneath this first-order isomorphism with Euclidean frameworks lies a stronger isomorphism between the inversive theory of m-circles constrained by angles in the plane and the congruence theory of planes in hyperbolic 3-space constrained by (hyperbolic) angles—or, dually, the theory of ideal points in hyperbolic space, constrained by hyperbolic distance. This correspondence applies beyond the first order to all levels of rigidity and flexibility and is presented in [21]. These isomorphic theories then share a common first-order isomorphism with Euclidean and spherical frameworks, which is explored in detail in [21]. While some aspects of this correspondence of plane circles and hyperbolic geometry are well known, there remain new insights to be extracted. It may be a surprise to people in CAD programming that simple plane and 3-space CAD contains, as a subset, the theory of planes and hyperplanes with angle constraints in hyperbolic 3- and 4-space.

This study continues a series of papers which investigate constraints among points, lines, and circles in the plane [2, 21, 22, 33]. One central problem is appropriate algorithms for "generic" behavior of configurations of objects and constraints. The
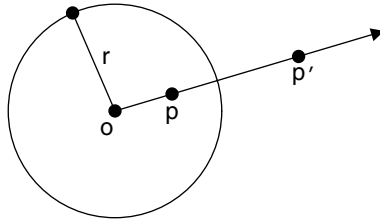
FIG. 2.1. *Inversion in a circle: $\vec{op} \cdot \vec{op}' = r^2$.*

results in this paper emphasize that there are plane problems that, unlike simple distance constraints in the plane, have a complexity that may not be polynomial. We offer further observations and unsolved problems in section 10.

2. Circles in the inversive plane. While people familiar with hyperbolic geometry and inversive geometry will be familiar with all the material in this section, we feel it is important to make this paper more self-contained for workers in CAD programming.

2.1. The plane model. Plane inversive geometry is the underlying geometry in this investigation, so this section will recall the definition and some useful results. Our two primary sources for these sections are [18, 34]. While this paper will focus on the plane, we will indicate how the results generalize directly to n dimensions $n \geq 1$ in section 8. The interested reader can find a nice treatment of inversive geometry in n-dimensional space in [34].

Let C be a circle centered at an arbitrary point o with radius r and p a point different from o. Then to the point p associate the point p' on the ray $\vec{op}$ satisfying $\vec{op} \cdot \vec{op}' = r^2$ (Figure 2.1). One easily finds that this association is a bijection of the plane onto itself, except for the point o. To complete the bijection, we adjoin exactly one point to the plane, the *point at infinity*, denoted by o', and pair it with o under this correspondence. The plane augmented with the point at infinity is the *inversive plane*, denoted Π.

DEFINITION 2.1. *Let C be a circle centered at a point o with radius r, o' the point at infinity, and p a point in the inversive plane. Then the* inverse of p in C is

1. *o' if $p = o$,*
2. *o if $p = o'$,*
3. *the point p' on the ray $\vec{op}$ satisfying $\vec{op} \cdot \vec{op}' = r^2$ otherwise.*

The point o is called the center of inversion *and C the* circle of inversion.

Note the points of the circle C are invariant under inversion in C.

2.2. The sphere model. This section describes the sphere model of the inversive plane. We include the sphere model because it sometimes provides a more convenient image for results about inversive geometry.

DEFINITION 2.2. *Let Σ be a sphere, p a point on Σ, and C a circle on Σ. Let t_C be the tip of the cone tangent to Σ at C. Then the* inverse of p in C is the second point of intersection of the line through p and t_C with Σ (Figure 2.2).

If the circle C in the above definition is a great circle (the intersection of a plane through the center of Σ with Σ), then t_C is a "point at infinity" in 3-space, and all lines through t_C and a point on Σ are parallel to the normal of the great circle. That
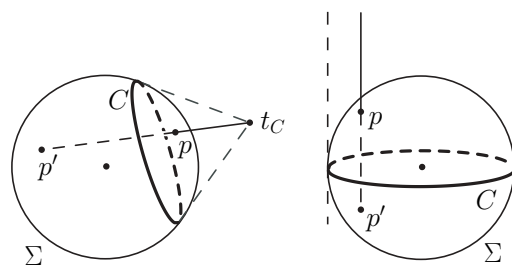
FIG. 2.2. *Inversion in a circle on the sphere model.*
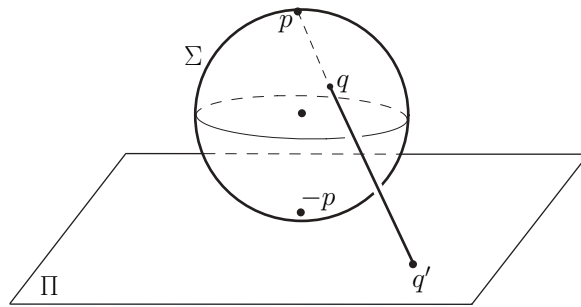
FIG. 2.3. *Stereographic projection from the sphere to the inversive plane.*

is, the inverse of a point p in a great circle is obtained by reflecting p in the great circle. Notice that in this spherical model the bijection of the sphere onto itself is complete; there is no need to augment the sphere with an additional point.

2.3. Connecting the plane model and the sphere model. The plane and sphere models of the inversive plane are connected through *stereographic projection*. Let p be a point on the sphere Σ and $-p$ the antipodal point of p. Let Π be a plane tangent to Σ at $-p$. Then we have a bijection between Σ and Π: $q \in \Sigma$ corresponds to $q' \in \Pi$, where q' is the point of intersection of the line pq with Π, and p corresponds to the point at infinity of Π (Figure 2.3).

Hence, the plane model Π of the inversive plane is obtained from the sphere Σ model by stereographic projection of the sphere from a point p onto a plane tangent to the sphere at the $-p$. Similarly, the sphere model is obtained from the plane model by *lifting* the plane onto the sphere (the inverse mapping of stereographic projection). Note that the point p of stereographic projection corresponds to the point at infinity in the plane model. Also note that circles through the projection point go to lines in the plane. Figure 2.4 illustrates that stereographic projection preserves angles of intersection of circles.

Notice that inversion in a great circle through the north and south poles is a reflection in the vertical plane of this great circle. This corresponds to an "inversion" in the projected line in the plane—now seen as a reflection in the line. Since we had not previously defined inversion in a line, we define this inversion as the reflection. Since all isometries of the plane are products of reflections, all isometries of the plane can be viewed as products of inversions.
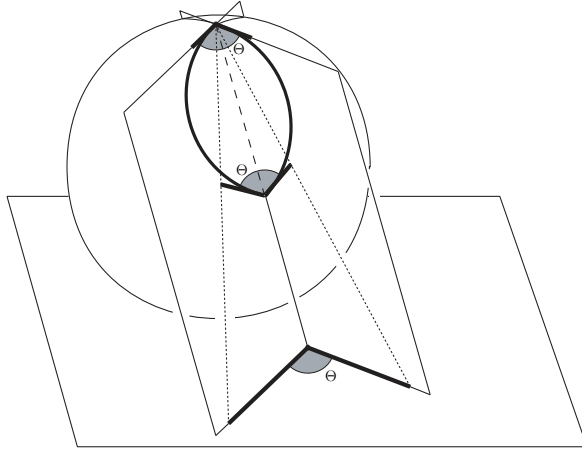
FIG. 2.4. *Stereographic projection preserves angles. The angles on the sphere are equal by symmetry. The vectors at the north pole are in a plane parallel to the inversive plane, so a translation gives the equality with the angle in the plane.*

2.4. Properties of inversions. The choice of the size of the sphere and the point of tangency of the sphere with the plane in the stereographic projection from the previous section is arbitrary. However, there is one such choice that makes many important facts about inversions visually obvious. Suppose C is the circle of inversion in the inversive plane Π centered at o with radius r. Take a sphere Σ of diameter r tangent to Π at o. Lifting Π onto Σ maps the circle of inversion onto the "equator" of the sphere—a great circle. Also, the point at infinity in the inversive plane maps onto the point of projection p on the sphere. The inversion in the sphere model is merely a reflection in this equatorial great circle. Stereographic projection of the reflected sphere back onto a plane through the great circle yields the image of the plane Π under the inversion.

This approach to inversion lends itself well to proving properties of inversions in the plane.

PROPOSITION 2.3. *Properties of inversions.*
(1) *The product of two inversions in the same circle is the identity mapping.*
(2) *Inversion preserves angles.*
(3) *The circle of inversion is invariant under inversion.*
(4) *Lines through the center of inversion are invariant under inversion.*
(5) *Circles orthogonal to the circle of inversion are invariant under inversion.*
(6) *The inverse of a circle through the center of inversion is a line not through the center of inversion.*
(7) *The inverse of a line not through the center of inversion is a circle through the center of inversion.*
(8) *The inverse of a circle not through the center of inversion is a circle not through the center of inversion.*

Proof. (1) follows from the fact that the product of two reflections of the sphere in the same great circle is the identity mapping.

(2). Since stereographic projection and reflections on the sphere preserve angles, it follows that inversion preserves angles.

(3), (4), (5). The mentioned objects lift onto objects invariant under the reflection in the "equator" of the sphere. Therefore, they are invariant under an inversion.
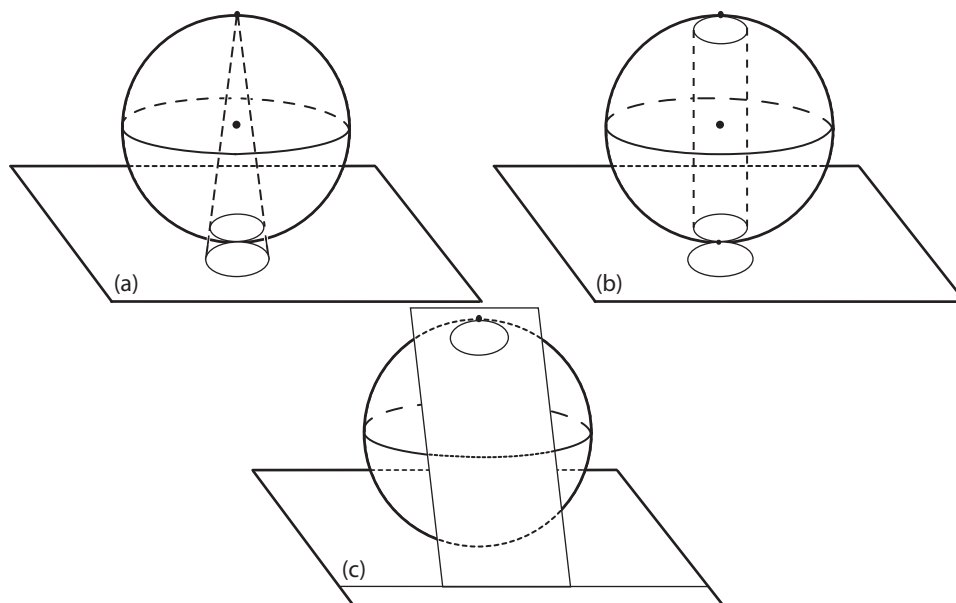
FIG. 2.5. *Proof without words of* (6) *and* (7) *of Proposition* 1: (a) *stereographic lifting of the circle;* (b) *inversion of the lift in the equator;* (c) *stereographic projection of the image.*

(6), (7). See Figure 2.5 for a proof without words of (6) and (7).

(8) follows from (6) and (7) and the fact that stereographic projection carries circles on the sphere onto circles on the plane.   □

The following section develops some of these properties of inversions algebraically.

2.5. An algebraic look at inversion. Since we will be working with circles and lines simultaneously, we introduce the following definition.

DEFINITION 2.4. *A* Möbius-Circle *(or* m-circle*) in the plane is a line or circle in the plane.*

Inversion can now be described as a map that carries m-circles to m-circles. For an algebraic representation of m-circles we begin with the equation

$$M \equiv a(x^2 + y^2) - 2bx - 2cy + d = 0,$$

where one of a, b, or c is nonzero ($a^2 + b^2 + c^2 \neq 0$). If $a \neq 0$, then M is a circle; if $a = 0$, then M is a line.

To study the inverse of an m-circle, take the circle of inversion to be the circle centered at the origin of radius k. Then the inverse of the point (x, y) is (x', y'), where

$$x' = k^2 \frac{x}{x^2 + y^2}, \quad y' = k^2 \frac{y}{x^2 + y^2}.$$

The inverse of the m-circle $ax^2 + ay^2 - 2bx - 2cy + d = 0$ is

$$d(x^2 + y^2) - 2bk^2 x - 2ck^2 y + ak^4 = 0.$$

The following is now obvious: the inverse of a circle through the center of inversion ($a \neq 0$ and $d = 0$) is a line not through the center of inversion; the inverse of a line through the center of inversion ($a = 0$ and $d = 0$) is a line through the center of
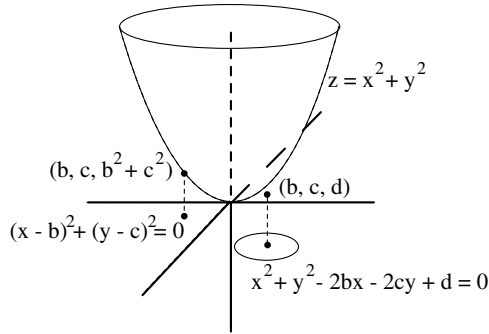
$$z = x^2 + y^2$$

$(b, c, b^2 + c^2)$

(b, c, d)

$(x - b)^2 + (y - c)^2 = 0$

$x^2 + y^2 - 2bx - 2cy + d = 0$

FIG. 3.1. *Representation of circles as points in* $\mathbb{R}^3$.

inversion; the inverse of a line not through the center of inversion ($a = 0$, $d \neq 0$) is a circle through the center of inversion.

An inversion in an arbitrary circle with the center at (x_0, y_0) is obtained by translating the center of inversion to the origin, performing an inversion about the origin, and translating back to (x_0, y_0).

3. Representation of circles as points in $\mathbb{R}^3$. We introduce a third model for m-circles in the inversive plane that will be central to our analysis (Figure 3.1).

3.1. Representation of circles as points in $\mathbb{R}^3$. Properties of a circle may be read from the representation

$$M \equiv a(x^2 + y^2) - 2bx - 2cy + d = 0.$$

If $a \neq 0$, the center of the circle M is $(\frac{b}{a}, \frac{c}{a})$, and the square of the radius is $\frac{b^2 + c^2 - ad}{a^2}$. If $a = 0$ and $c \neq 0$, then the slope of line M is given by $-\frac{b}{c}$ and the y-intercept by $\frac{d}{2c}$. If $c = 0$, then $b \neq 0$, and the line is $x = \frac{d}{2b}$.

We can represent M by the four-tuple (a, b, c, d). These are homogeneous coordinates of M since $(\lambda a, \lambda b, \lambda c, \lambda d)$ represents the same m-circle as (a, b, c, d) for nonzero λ. It is convenient to normalize to make the coordinates unique.

One choice would be to set $d = 1$. This is equivalent to insisting that the circles (and lines) do not pass through the origin, since the origin is not a point on any circle with $d = 1$. Every m-circle in the plane not through the origin would be represented by a unique four-tuple $(a, b, c, 1)$ such that $(a, b, c) \neq (0, 0, 0)$.

Instead we will take a second normalization, $a = 1$. So now we are dealing directly with circles only, with the circles through the origin corresponding to lines by an inversion in the unit circle. This representation of a circle now takes the form of a triple (b, c, d), insisting that $a = 1$. The center of a circle is (b, c), and the square of the radius is $b^2 + c^2 - d$. This normalization is the inverse of the normalization $d = 1$ above. That is, we are omitting all the circles through the point at infinity (i.e., lines).

In [18] Pedoe presents a nice three-dimensional representation of this model. The three-tuple (b, c, d) corresponds to the circle centered at (b, c) of radius $\sqrt{b^2 + c^2 - d}$, so given a point in $\mathbb{R}^3$, the center of the corresponding circle is the vertical projection of that point onto the xy-plane. If Ω denotes the paraboloid $z = x^2 + y^2$, then a point on Ω represents a circle with radius zero (a point circle), a point above Ω represents a circle with complex radius, and a point below Ω represents a circle with a positive
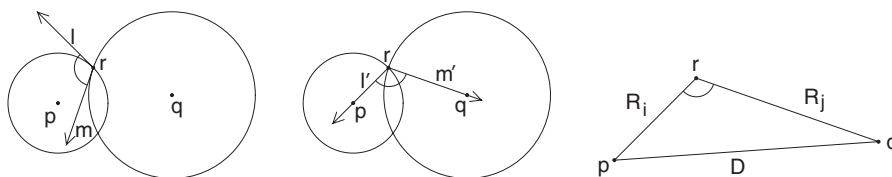
FIG. 3.2. *Measuring the angle of intersection of two circles: rotate by* $90°$ *and form a triangle with the centers.*

radius. Since we are primarily interested in circles with positive radius, we will focus on the points outside the paraboloid. We refer to this model as the *paraboloid model of the inversive plane.*

3.2. The angle of intersection between two circles. In order to eliminate the ambiguity of the angle of intersection between two circles, we introduce a convention. We orient circles in the counterclockwise direction and measure the angle between the oriented tangents to the circles at the point of intersection. In Figure 3.2, the angle of intersection of the two circles is the angle subtended by l and m in the counterclockwise direction. A rotation of $\frac{\pi}{2}$ sends l, m onto l', m'. Since l, m are tangents to the circles, l', m' pass through the centers of the circles. Therefore, the angle of intersection is $\angle prq$, where p and q are the centers of the circles and r is the point of intersection. (Note that the angle at the second point of intersection of the circles is identical.)

The cosine law applied to the triangle prq in Figure 3.2 yields

$$(3.1) \qquad cos(\angle prq) = \frac{R_i^2 + R_j^2 - D^2}{2R_iR_j},$$

where R_i, R_j are the radii of the circles, and D is the distance $|pq|$. Representing the circles by points (b_i, c_i, d_i) and (b_j, c_j, d_j) in $\mathbb{R}^3$, (3.1) becomes

$$(3.2) \qquad \mathcal{K}_{ij} \equiv cos(\angle prq) = \frac{2b_ib_j + 2c_ic_j - d_i - d_j}{2\sqrt{(b_i^2 + c_i^2 - d_i)(b_j^2 + c_j^2 - d_j)}}.$$

As a special case, two circles (b_i, c_i, d_i) and (b_j, c_j, d_j) are orthogonal iff

$$(3.3) \qquad 2b_ib_j + 2c_ic_j - d_i - d_j = 0.$$

This condition is equivalent to $R_i^2 + R_j^2 = D^2$.

3.3. Coaxal systems and bundles of circles. There is a rich geometry of circles that would be needed to explore examples in these designs and to explore special position configurations. However, for the specific content of this paper, we will just briefly describe two specific families related to linear dependence of m-circles.

Two distinct circles $\mathbf{m}_1 = (b_1, c_1, d_1)$ and $\mathbf{m}_2 = (b_2, c_2, d_2)$ span the family of circles called the *coaxal system of circles generated by* $\mathbf{m}_1$ *and* $\mathbf{m}_2$, obtained by taking affine combinations of $\mathbf{m}_1$ and $\mathbf{m}_2$,

$$\lambda\mathbf{m}_1 + (1 - \lambda)\mathbf{m}_2 = (\lambda b_1 + (1 - \lambda)b_2, \lambda c_1 + (1 - \lambda)c_2, \lambda d_1 + (1 - \lambda)d_2).$$

Therefore, a coaxal system of circles is represented by a line l in $\mathbb{R}^3$. This line projects onto a line in the xy-plane; hence the centers of the circles in the coaxal system are
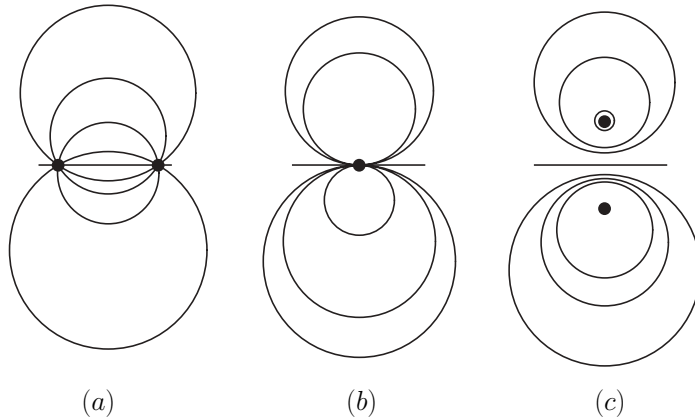
FIG. 3.3. *Coaxal system of circles:* (a) *where the line missed the paraboloid;* (b) *where the line is tangent to the paraboloid;* (c) *where the line intersects the paraboloid.*
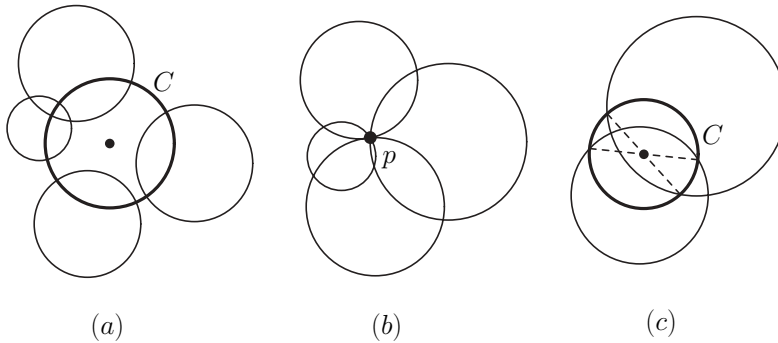

FIG. 3.4. *Bundles of circles:* (a) *circles orthogonal to the fixed circle C;* (b) *circles through the fixed point p;* (c) *circles intersecting antipodal points of the fixed circle C or, equivalently, circles orthogonal to a circle with a complex radius.*

collinear. Visually and algebraically, there are three different types of coaxal systems of circles corresponding to whether l misses, is tangent to, or intersects the paraboloid Ω. The three types of coaxal systems of circles are illustrated in Figure 3.3.

A *bundle of circles* is generated by three affinely independent circles, represented by a plane $\mathbf{P}$ in $\mathbb{R}^3$. There are three types of bundles depending on whether $\mathbf{P}$ misses, is tangent to, or intersects the paraboloid $z = x^2 + y^2$. The three types of bundles are illustrated in Figure 3.4.

4. m-circle designs. An *m-circle design* $(G, \mathbf{m})$ is a graph $G = (V, E)$ together with a point $\mathbf{m} \in \mathbb{R}^{3|V|}$, where $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_i, \ldots, \mathbf{m}_{|V|})$, $i \in V$, such that $b_i^2 + c_i^2 - d_i > 0$ for each $\mathbf{m}_i = (b_i, c_i, d_i)$. We wish to track when two m-circle designs are equivalent under inversion, but that problem is too difficult. Instead we will consider a simpler problem: When is the design $(G, \mathbf{m})$ unique, with the given angles, in a neighborhood of $\mathbf{m}$ in $\mathbb{R}^{3|V|}$? In full generality, this local uniqueness is also too hard—but it does have a linearized version that answers the question of local uniqueness for almost all designs $\mathbf{m} \in \mathbb{R}^{3|V|}$. This linearized or *first-order* version is studied in the next few subsections. We will then return to state the standard results about how this first-order analysis demonstrates the local uniqueness.

4.1. The constraint function. The *constraint function* $\mathcal{K}_{i,j}$ for two circles $\mathbf{m}_i$ and $\mathbf{m}_j$ of nonzero radius is

$$(4.1) \qquad \mathcal{K}_{i,j} = \frac{2b_i b_j + 2c_i c_j - d_i - d_j}{2\sqrt{(b_i^2 + c_i^2 - d_i)(b_j^2 + c_j^2 - d_j)}}.$$

Note that the constraint function has an obvious geometric interpretation only if $\mathcal{K}_{i,j} \in [-1,1]$—it measures the cosine of the angle of intersection between the circles $\mathbf{m}_i$ and $\mathbf{m}_j$. However, the constraint function exists for nonintersecting circles as well and can be used for geometric purposes. It takes the value $\cosh \delta$, where δ is the natural logarithm of the ratio (larger to smaller) of the radii of two concentric circles. It can be shown that any two nonintersecting circles can be inverted into two concentric circles and that this ratio is constant. δ is called the *inversive distance* between the two circles [34].

In general, a single inversion in a circle multiplies the value of $\mathcal{K}_{i,j}$ by -1. Since inversion preserves angles and inversive distance, $|\mathcal{K}_{i,j}|$ is invariant under all inversions.

However, for local uniqueness, we will restrict ourselves to the group of *direct circular transformations*: products of an even number of inversions. In general, a single inversion is not local—it takes a configuration to a "faraway" configuration in the appropriate metric for configurations in $\mathbb{R}^3$. This group includes translations, rotations, dilations by a positive factor, etc. The value of $\mathcal{K}_{i,j}$ is invariant under this group for all circles with positive radii. In fact, the constraint function is invariant even for circles with imaginary radii. The function is not defined for point circles (of radius 0).

4.2. Shakes and the constraint matrix. Let $\mathbf{m}_i = (b_i, c_i, d_i)$ and $\mathbf{m}_j = (b_j, c_j, d_j)$ be two m-circles, with nonzero radius and constraint $\mathcal{K}_{i,j} = C$, where C is some constant. If $\mathbf{m}(t) = (\mathbf{m}_i(t), \mathbf{m}_j(t))$ is a path differentiable at $t = 0$ with $\mathbf{m}(0) = (\mathbf{m}_i, \mathbf{m}_j)$, then

$$(4.2) \qquad \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_{i,j}\right)(0) = \frac{\mathbf{k}_{i,j}}{h} \cdot \mathbf{m}_i' + \frac{\mathbf{k}_{j,i}}{h} \cdot \mathbf{m}_j' = 0,$$

where

$$\frac{\mathbf{k}_{i,j}}{h} = \left[\frac{\partial}{\partial b_i}\mathcal{K}_{i,j}, \frac{\partial}{\partial c_i}\mathcal{K}_{i,j}, \frac{\partial}{\partial d_i}\mathcal{K}_{i,j}\right],$$
$$\mathbf{m}_i' = \left[\frac{\mathrm{d}}{\mathrm{d}t}b_i, \frac{\mathrm{d}}{\mathrm{d}t}c_i, \frac{\mathrm{d}}{\mathrm{d}t}d_i\right],$$
$$h = 2\sqrt{(b_i^2 + c_i^2 - d_i)(b_j^2 + c_j^2 - d_j)}.$$

Since $h \neq 0$, (4.2) is equivalent to

$$(4.3) \qquad \mathbf{k}_{i,j} \cdot \mathbf{m}_i' + \mathbf{k}_{j,i} \cdot \mathbf{m}_j' = 0.$$

This prompts the following definition.

DEFINITION 4.1. *Let $(G, \mathbf{m})$ be an m-circle design. The vector $\mathbf{m}' \in \mathbb{R}^{3|V|}$ is a first-order motion or* shake *of $(G, \mathbf{m})$ if for every $\{i, j\} \in E$,*

$$\mathbf{k}_{i,j} \cdot \mathbf{m}_i' + \mathbf{k}_{j,i} \cdot \mathbf{m}_j' = 0.$$

This system of linear equations generates the constraint matrix $\mathcal{C}(G, \mathbf{m})$ of the m-circle design $(G, \mathbf{m})$. *The space of shakes of* $(G, \mathbf{m})$ *is precisely the nullspace of* $\mathcal{C}(G, \mathbf{m})$.

Remark. The constraint matrix has one row for each constraint in E and three columns for each m-circle $\mathbf{m}_i$ (one column for each of b_i, c_i, d_i). So $\mathcal{C}(G, \mathbf{m})$ is a $|E| \times 3|V|$ matrix. For reference we note that $\mathbf{k}_{i,j} = [h\frac{\partial}{\partial b_i}\mathcal{K}_{i,j}, h\frac{\partial}{\partial c_i}\mathcal{K}_{i,j}, h\frac{\partial}{\partial d_i}\mathcal{K}_{i,j}]$, where

$$h\frac{\partial}{\partial b_i}\mathcal{K}_{i,j} = \frac{b_i d_j + 2b_j c_i^2 - 2b_j d_i - 2b_i c_i c_j + b_i d_i}{b_i^2 + c_i^2 - d_i},$$

$$h\frac{\partial}{\partial c_i}\mathcal{K}_{i,j} = \frac{c_i d_j + 2c_j b_i^2 - 2c_j d_i - 2c_i b_i b_j + c_i d_i}{b_i^2 + c_i^2 - d_i},$$

$$h\frac{\partial}{\partial d_i}\mathcal{K}_{i,j} = -\frac{1}{2}\frac{2b_i^2 + 2c_i^2 - 2b_i b_j - 2c_i c_j - d_i + d_j}{b_i^2 + c_i^2 - d_i},$$

and

$$\mathcal{K}(G, \mathbf{m}) = \begin{array}{c} \\ \{1,2\} \\ \\ \{i,j\} \\ \\ \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & \cdots & i & \cdots & j & \cdots & v \end{matrix} \\ \begin{pmatrix} \mathbf{k}_{1,2} & \mathbf{k}_{2,1} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{k}_{i,j} & \cdots & \mathbf{k}_{j,i} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} \end{array}.$$

4.3. The trivial shakes. The following six vectors, called the *trivial shakes*, are in the nullspace of $\mathcal{C}(G, \mathbf{m})$ for any graph G on the given circles $\mathbf{m}$.

$$(4.4) \quad \begin{bmatrix} b_1^2 - \frac{1}{2}d_1 \\ b_1 c_1 \\ b_1 d_1 \\ \vdots \\ b_v^2 - \frac{1}{2}d_v \\ b_v c_v \\ b_v d_v \end{bmatrix}, \begin{bmatrix} b_1 c_1 \\ c_1^2 - \frac{1}{2}d_1 \\ c_1 d_1 \\ \vdots \\ b_v c_v \\ c_v^2 - \frac{1}{2}d_v \\ c_v d_v \end{bmatrix}, \begin{bmatrix} \frac{1}{2}c_1 \\ -\frac{1}{2}b_1 \\ 0 \\ \vdots \\ \frac{1}{2}c_v \\ -\frac{1}{2}b_v \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -b_1 \\ \vdots \\ -\frac{1}{2} \\ 0 \\ -b_v \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -c_1 \\ \vdots \\ 0 \\ -\frac{1}{2} \\ -c_v \end{bmatrix}, \begin{bmatrix} -b_1 \\ -c_1 \\ -2d_1 \\ \vdots \\ -b_v \\ -c_v \\ -2d_v \end{bmatrix}.$$

These six vectors are linearly independent provided $v = |V| \geq 3$ and at least three circles do not lie in a coaxal system.

Remark. The six solution vectors described above correspond to instantaneous velocities experienced by the m-circle design at the beginning of each type of even inversive transformation. The first two vectors correspond to translations in the x- and y-directions, respectively. The third vector corresponds to a rotation about the origin. The fourth and fifth vectors correspond to a reflection in an axis followed by an inversion in a circle tangent to that axis. The sixth vector corresponds to the product of two inversions in concentric circles—a dilation.

4.4. Stiff m-circle designs. A configuration $\mathbf{m}$ of m-circles is in *general position* if no set of 4 of the m-circles lie in a bundle, and if $|V| = 3$, then the circles are not coaxal. An m-circle design $(G, \mathbf{m})$ with $|V| \geq 3$ and $\mathbf{m}$ in general position is *inversively stiff*, or just *stiff*, if the kernel of $\mathcal{C}(G, m)$ is generated by the trivial shakes
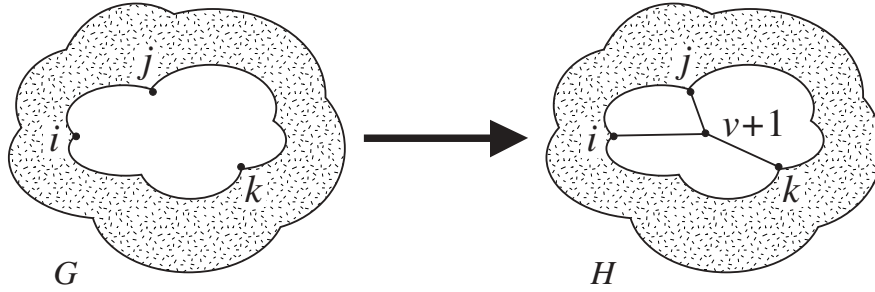
FIG. 4.1. *Extending the graph G of a stiff design to a graph H for a larger stiff design.*

$(\operatorname{rank}(\mathcal{C}(G, m)) = 3|V| - 6)$. Otherwise, $(G, \mathbf{m})$ is said to be *inversively shaky*, or just *shaky*.

This section proves the following characterization of stiff m-circle designs (see Theorem 4.5): An m-circle design $(G, \mathbf{m})$ with $|V| \geq 3$ and $\mathbf{m}$ in general position is stiff iff the nullspace of $\mathcal{C}(G, \mathbf{m})$ is equal to the nullspace of $\mathcal{C}(K_{|V|}, \mathbf{m})$, where $K_{|V|}$ is the complete graph on $|V|$ vertices.

LEMMA 4.2. *If $|V| \geq 3$, then $\operatorname{rank}(\mathcal{C}(G, \mathbf{m})) \leq 3|V| - 6$.*

Proof. If we have at least three inversively independent circles, then the assertion follows directly from the independence of the six trivial motions (see (4.4)). If all of the m-circles are dependent on two m-circles, then it is a simple matter to see that the rank of the matrix can drop only from the maximum dimension achieved for independent m-circles. In fact, in this case, the rank becomes $|V| - 1 < 3|V| - 6$. □

LEMMA 4.3. *For $|V| \geq 3$ and $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_{|V|})$ in general position, there exists a graph $G = (V, E)$ such that $\operatorname{rank}(\mathcal{C}(G, \mathbf{m})) = 3|V| - 6$.*

Proof. The proof will induct on the number of vertices of G. For $|V| = 3$, take the design $(G, \mathbf{m})$, where G is the complete graph on three vertices K_3 (a triangle) and $\mathbf{m}_1 = (1, 0, 0)$, $\mathbf{m}_2 = (0, 1, 0)$, $\mathbf{m}_3 = (0, 0, 1)$. Note that $\mathbf{m}_1$, $\mathbf{m}_2$, and $\mathbf{m}_3$ are in general position. The constraint matrix for $(G, \mathbf{m})$ is

$$\mathcal{C}(G, \mathbf{m}) = \begin{bmatrix} 0 & 2 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -3/2 & 0 & 0 & 0 & 2 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -3/2 & 0 & 2 & -1/2 \end{bmatrix}.$$

The second, seventh, and eighth columns of $\mathcal{C}(G, \mathbf{m})$ yield a 3×3 matrix of rank 3, so $\mathcal{C}(G, \mathbf{m})$ has rank $3 = 3|V| - 6$.

Suppose there exists a graph G with v vertices such that $\operatorname{rank}(\mathcal{C}(G, \mathbf{m})) = 3v - 6$. Let H be the graph obtained from G by adding a new vertex $v + 1$ to G and three new edges, each connecting $v+1$ to the distinct vertices i, j, k of G (Figure 4.1). Therefore, a new circle $\mathbf{m}_{v+1}$ is added to the design $(G, \mathbf{m})$ with three distinct constraints, creating the new design $(H, \mathbf{n})$. So if the constraint matrix of $(G, \mathbf{m})$ is

$$\mathcal{C}(G, \mathbf{m}) = \begin{pmatrix} \mathbf{k}_{1,2} & \mathbf{k}_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & \vdots & & \vdots & & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{f,g} & \cdots & \mathbf{k}_{g,f} \end{pmatrix},$$

then the constraint matrix $\mathcal{C}(H, \mathbf{n})$ is

$$
\begin{array}{c}
\\
\\
\\
\\
\\
\{i,v+1\} \\
\{j,v+1\} \\
\{k,v+1\}
\end{array}
\left(
\begin{array}{cccccccc|c}
& i & & j & & k & & & v+1 \\
\mathbf{k}_{1,2} & \mathbf{k}_{2,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & & \vdots & & & \vdots & & \vdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{f,g} & \cdots & \mathbf{k}_{g,f} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{k}_{i,v+1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{k}_{v+1,i} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{j,v+1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{k}_{v+1,j} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{k,v+1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{k}_{v+1,k}
\end{array}
\right),
$$

where $\mathbf{n} = (\mathbf{m}, \mathbf{m}_{v+1})$. The three added columns account for the m-circle $\mathbf{m}_{v+1}$, and the three added rows account for the constraint equations. Now, $\mathrm{rank}(\mathcal{C}(H, \mathbf{n})) = 3(v+1) - 6 = 3v - 3$ iff the three new rows of $\mathcal{C}(H, \mathbf{n})$ add 3 to the rank of $\mathcal{C}(G, \mathbf{m})$. The rank of $\mathcal{C}(G, \mathbf{m})$ increases by 3 iff

$$
\left|
\begin{array}{c}
\mathbf{k}_{v+1,i} \\
\mathbf{k}_{v+1,j} \\
\mathbf{k}_{v+1,k}
\end{array}
\right| \neq 0.
$$

This condition is equivalent to

$$
\frac{2}{r_{v+1}^2}
\left|
\begin{array}{cccc}
b_{v+1} & c_{v+1} & d_{v+1} & 1 \\
b_i & c_i & d_i & 1 \\
b_j & c_j & d_j & 1 \\
b_k & c_k & d_k & 1
\end{array}
\right| \neq 0,
$$

where r_{v+1} is the radius of the circle $\mathbf{m}_{v+1}$. Therefore, unless $\mathbf{m}_{v+1}$ is a linear combination of the circles $\mathbf{m}_i$, $\mathbf{m}_j$, and $\mathbf{m}_k$, that is, unless $\mathbf{m}_i$, $\mathbf{m}_j$, $\mathbf{m}_k$, and $\mathbf{m}_{v+1}$ lie in the same bundle, $\mathrm{rank}(\mathcal{C}(H, \mathbf{n})) = 3v - 3$. By assumption, the m-circle design is in general position, so this determinant is nonzero, and the new design has rank $3(|V| + 1) - 6$ as required. $\square$

LEMMA 4.4. *For $|V| \geq 3$ and $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_{|V|})$ in general position, the nullspace of $\mathcal{C}(K_{|V|}, \mathbf{m})$ is generated by the trivial shakes.*

Proof. Let $\mathcal{S}$ denote the span of the trivial shakes. Since the trivial shakes are solutions to the linear system $\mathcal{C}(G, \mathbf{m})\mathbf{m}' = 0$ for any graph G, we have that $\mathcal{S} \subset \mathrm{null}(\mathcal{C}(G, \mathbf{m}))$.

By Lemma 4.3, for any $|V| \geq 3$ and any $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_{|V|})$ in general position, there exists a graph $G = (V, E)$ with $\mathrm{rank}(\mathcal{C}(G, \mathbf{m})) = 3|V| - 6$. Add edges to G to obtain the complete graph $K_{|V|}$ on the vertex set V. Therefore,

$$
3|V| - 6 = \mathrm{rank}(\mathcal{C}(G, \mathbf{m})) \leq \mathrm{rank}(\mathcal{C}(K_{|V|}, \mathbf{m})),
$$

and Lemma 4.2 gives

$$
\mathrm{rank}(\mathcal{C}(K_{|V|}, \mathbf{m})) \leq 3|V| - 6.
$$

Therefore, $\mathrm{rank}(\mathcal{C}(K_{|V|}, \mathbf{m})) = 3|V| - 6$ and

$$
\dim \mathrm{null}(\mathcal{C}(G, \mathbf{m})) = 3|V| - \mathrm{rank}(\mathcal{C}(K_{|V|}, \mathbf{m})) = 6 = \dim \mathcal{S}.
$$

Therefore, $\mathrm{null}(\mathcal{C}(G, \mathbf{m})) = \mathcal{S}$. $\square$

THEOREM 4.5. *An m-circle design $(G, \mathbf{m})$ with $|V| \geq 3$ and $\mathbf{m}$ in general position is stiff iff the nullspace of $\mathcal{C}(G, \mathbf{m})$ is equal to the nullspace of $\mathcal{C}(K_{|V|}, \mathbf{m})$, where $K_{|V|}$ is the complete graph on $|V|$ vertices.*

4.5. Stiffness and local uniqueness. Our original goal was to study whether an m-circle design $(G, \mathbf{m})$ was locally unique, up to inversions. Explicitly, we have a map $f_G : \mathbb{R}^{3|V|} \to \mathbb{R}^{|E|}$ that measures the "cosine of the angle" or the value of $\mathcal{K}_{i,j}$ for each edge in the graph: $f(\mathbf{m}) = (\ldots, k_{i,j}, \ldots)$. Let $I(\mathbf{m})$ be the set of all configurations equivalent to $\mathbf{m}$ by inversions.

DEFINITION 4.6. *A design* $(G, \mathbf{m})$ *is* locally unique *if there is an open neighborhood* $N_{\mathbf{m}}$ *of* $\mathbf{m}$ *such that* $f_G^{-1} f_G(\mathbf{m}) \cap N_{\mathbf{m}} \subset I(\mathbf{m})$.

Our constraint matrix is then the Jacobian $\mathrm{d}f_G$ of the function f_G. Moreover, the function f_G is (up to squaring entries) a polynomial function. There are standard results about how this Jacobian at $\mathbf{m}$ predicts the dimension of the space $f_G^{-1}(f_G(\mathbf{m})) \cap N_{\mathbf{m}}$, provided the point is *regular*, that is, that the Jacobian achieves its maximum rank at the point $\mathbf{m}$ [4, 20].

For polynomial functions, there are also standard results that state the failure of local uniqueness is equivalent to the existence of an analytic path $\mathbf{m}(t)$, $0 \le t < 1$, of inversively inequivalent configurations inside $f_G^{-1}(f_G(\mathbf{m}))$ (that is, all constraints are the same as at $\mathbf{m}$, but some other value of $\mathcal{K}_{h,l}$ is changing for some $\{h, l\} \notin E$). The following is a translation of the analogous result for bar-and-joint frameworks [4, 20]. We say that an m-circle design is *flexible* if there is an analytic path $\mathbf{m}(t)$, $0 \le t \le 1$ of inversively equivalent designs with the same constraints.

THEOREM 4.7. *If an m-circle design* $(G, \mathbf{m})$ *is stiff, then the design is locally unique. If* $(G, \mathbf{m})$ *is never stiff for any m-circle configuration, then* $(G, \mathbf{m})$ *is flexible for all regular points* $\mathbf{m}$ *at which* $\mathcal{C}(G, \mathbf{m})$ *achieves its maximum rank.*

Proof. For $|V| = 1$, all designs are stiff, and all single circles are inversively unique up to translations (of the center) and dilation (of the radius).

For $|V| = 2$, with distinct circles, there are two cases. If the edge is not present, then the design is not stiff, nor is the design even locally unique (e.g., change the distance between the centers without changing radii). However, if the edge is present, then all the solutions to $\mathcal{C}(G, \mathbf{m}) \times \mathbf{m}' = \mathbf{0}$ are trivial, and the design is stiff. In this case, any two circles with the same value of $\mathcal{K}_{1,2}$ are equivalent under inversion, and the design is, again, unique.

Assume that $(G, \mathbf{m})$ is stiff with $|V| \ge 3$. Therefore, the only inversive shakes are the trivial shakes that are derivatives of direct circular maps.

Assume we have an analytic path $\mathbf{m}(t)$ preserving the constraints; then this can be replaced by an analytic flex. If we take the first derivatives along this path, with the angles fixed, we will find a shake of the design. If this shake is not the derivative of an angle preserving map, then we know that the design was not stiff. This is a contradiction.

However, if a flex is not angle preserving, it may be the kth derivative that is not the derivative of an angle preserving map. By adding some angle preserving map, we can ensure that the first $k - 1$ derivatives of the flex are all zero. (For example, we can assume that an initial circle is fixed and that other circles are fixed as long as the derivatives match angle preserving maps.) With this assumption, it is easy to verify that the kth derivative of the constraint functions gives a (nontrivial) shake to the design. This is the desired contradiction.

On the other hand, assume that $(G, \mathbf{m})$ is never stiff for any m-circle configuration. The inverse function theorem guarantees that, at regular points, the dimension of $f_G^{-1}(f_G(\mathbf{m})) \cap N_{\mathbf{m}}$ is more than 6. There is a sequence $\mathbf{m}(n)$, $n = 1, 2, \ldots$ of designs converging to $\mathbf{m}$ that preserve the angle constraints. By the curve selection theorem of Milnor, which applies to the constraints, if there is such a converging sequence preserving these constraints (which can be written in polynomial form), then there

is a piecewise analytic path preserving the constraints [4, 20]. This gives the desired flex at the selected regular point.   □

5. Bar-and-joint frameworks in Euclidean 3-space. So far, our study of circles in the plane under angle constraints is clearly analogous to the study of bar-and-joint frameworks in Euclidean 3-space. The constraints for a bar-and-joint framework take the form of a distance constraint (a bar) between two vertices (joints) and generate graphs and constraint matrices for which the results precisely match the results of the previous sections [24, 25, 32]. We will briefly summarize this theory in order to present a precise isomorphism that underlies this analogy.

As before, we start with a graph $G = (V, E)$ and create a framework by realizing the vertices as points in 3-space.

DEFINITION 5.1. *A* bar-and-joint framework *or* framework $(G, \mathbf{p})$ *in* $\mathbb{R}^3$ *is a graph* $G = (V, E)$ *together with a configuration or point* $\mathbf{p} \in \mathbb{R}^{3|V|}$*, where* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_i, \ldots, \mathbf{p}_{|V|})$*,* $i \in V$*.*

5.1. First-order motions and the rigidity matrix. A *first-order motion* of the framework $(G, \mathbf{p})$ is a map $\mathbf{u} : V \to \mathbb{R}^3$, where we denote $\mathbf{u}(i)$ by $\mathbf{u}_i$, such that for every edge $\{i, j\} \in E$,

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0.$$

This gives rise to the *rigidity matrix* of the bar-and-joint framework $(G, \mathbf{p})$:

$$
R(G, \mathbf{p}) \;=\; \{i,j\}
\begin{pmatrix}
 & \overset{i}{\vdots} & \cdots & \overset{j}{\vdots} & \\
\cdots & \mathbf{p}_i - \mathbf{p}_j & \cdots & \mathbf{p}_j - \mathbf{p}_i & \cdots \\
 & \vdots & & \vdots &
\end{pmatrix}
$$

(with all other entries zero). The nullspace of the rigidity matrix is the space of first-order motions of $(G, \mathbf{p})$. A first-order motion of a framework is *trivial* if the motion is a restriction of the derivative of a Euclidean motion of $\mathbb{R}^3$, restricted to the vertices of the framework. The framework $(G, \mathbf{p})$ is *first-order rigid* if all the motions of $(G, \mathbf{p})$ are trivial.

5.2. Trivial solutions of the rigidity matrix. The following are six linearly independent vectors in the space of first-order motions for any framework in $\mathbb{R}^3$ with at least 3 noncollinear vertices. They correspond to translations in the x-, y-, and z-directions and rotations about the x-, y-, and z-axes, respectively:

$$
\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix},
\begin{bmatrix} 0 \\ -z_1 \\ y_1 \\ \vdots \\ 0 \\ -z_{|V|} \\ y_{|V|} \end{bmatrix},
\begin{bmatrix} -z_1 \\ 0 \\ x_1 \\ \vdots \\ -z_{|V|} \\ 0 \\ x_{|V|} \end{bmatrix},
\begin{bmatrix} -y_1 \\ x_1 \\ 0 \\ \vdots \\ -y_{|V|} \\ x_{|V|} \\ 0 \end{bmatrix}.
$$

A framework is first-order rigid if its space of first-order motions is generated by these motions.

6. An isomorphism between frameworks and m-circle designs. It can shown that a first-order rigid framework on $v \geq 3$ joints requires at least $3v - 6$ bars. The proof is identical to the proofs presented in section 4. In fact, the proofs from section 4 were adapted from proofs for the equivalent statements for bar-and-joint frameworks. Therefore, the counts for rigid bar-and-joint frameworks and stiff m-circle designs are identical. This is not a coincidence: there is a geometric isomorphism between the two first-order theories.

The key observation is the following identity relating the determinant of a sub-matrix of $\mathcal{C}(G, \mathbf{m})$ with the determinant of a submatrix of $R(G, \mathbf{m})$:

$$
\begin{vmatrix} \mathbf{k}_{n,i} \\ \mathbf{k}_{n,j} \\ \mathbf{k}_{n,k} \end{vmatrix} = \frac{2}{r_n^2} \begin{vmatrix} b_n & c_n & d_n & 1 \\ b_i & c_i & d_i & 1 \\ b_j & c_j & d_j & 1 \\ b_k & c_k & d_k & 1 \end{vmatrix} = \frac{2}{r_n^2} \begin{vmatrix} b_n - b_i & c_n - c_i & d_n - d_i \\ b_n - b_j & c_n - c_j & d_n - d_j \\ b_n - b_k & c_n - c_k & d_n - d_k \end{vmatrix}.
$$

This identity suggests the existence of a linear transformation carrying $\mathcal{C}(G, \mathbf{m})$ onto $R(G, \mathbf{m})$. Indeed, the system

$$
\begin{bmatrix} \mathbf{k}_{n,i} \\ \mathbf{k}_{n,j} \\ \mathbf{k}_{n,k} \end{bmatrix} \mathbf{T}_n = \begin{bmatrix} b_n - b_i & c_n - c_i & d_n - d_i \\ b_n - b_j & c_n - c_j & d_n - d_j \\ b_n - b_k & c_n - c_k & d_n - d_k \end{bmatrix}
$$

has the solution

$$
\mathbf{T}_n = \begin{bmatrix} -\frac{1}{2} & 0 & -b_n \\ 0 & -\frac{1}{2} & -c_n \\ -b_n & -c_n & -2d_n \end{bmatrix}.
$$

In general, $\mathcal{C}(G, \mathbf{m}) \, \mathbf{T_m} = R(G, \mathbf{p})$, where $\mathbf{T_m}$ is the block diagonal matrix, which depends only on the point $\mathbf{m}$,

$$
\mathbf{T_m} = \begin{bmatrix} \mathbf{T}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{T}_2 & \cdots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{T}_v \end{bmatrix}.
$$

We summarize this discussion.

THEOREM 6.1. *Given an m-circle design $(G, \mathbf{m})$, there is an invertible transformation $\mathbf{T_m}$ such that $\mathcal{C}(G, \mathbf{m}) \times \mathbf{T_m} = R(G, \mathbf{m})$. In particular, the m-circle design $(G, \mathbf{m})$ is stiff iff the bar-and-joint framework $(G, \mathbf{m})$ is first-order rigid.*

Notice that not all configurations in 3-space for frameworks are m-circle configurations, since we have restricted ourselves to circles of positive radius (points in $\mathbb{R}^3$ below the paraboloid).

Remark. It is important to note that the two theories are only infinitesimally equivalent. That is, on the level of Jacobians, the two theories are equivalent. There is no transformation that carries the constraint equation of one system into the constraint equation of the other system. Rigidity is not equivalent, but first-order rigidity is equivalent to stiffness. The regular points of the two maps for a fixed G are identical (they are defined by the rank of the isomorphic Jacobians). The rigidity or flexibility of frameworks or designs at these regular points (which form an open dense subset of $\mathbb{R}^{3v}$) are also equivalent. The distinctions all occur on the singular points of the two maps, where one configuration may still give a locally unique design for one map but a flexible framework for the other.
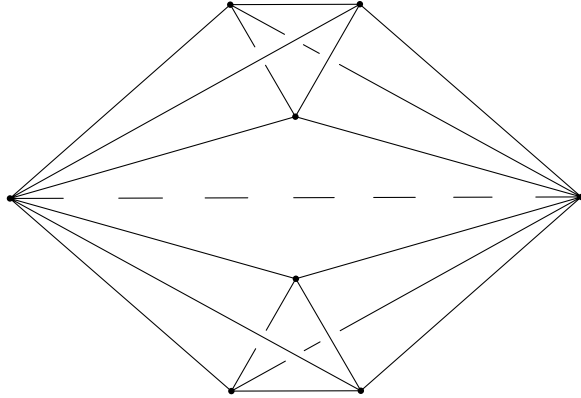
FIG. 7.1. *Although the framework satisfies all the conditions of Proposal* 1, *the framework is not rigid: there is a rotation of one half of the framework about the dotted line.*
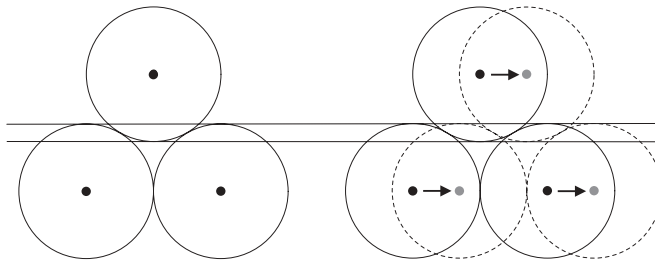

FIG. 7.2. *An m-circle design corresponding to the framework in Figure* 7.1.

7. Transferring between theories. There are many results and conjectures for the widely studied first-order theory of frameworks in 3-space that convert easily into the less studied theory of circles and angles in the inversive plane [9, 29, 30, 32]. However, it is important to emphasize that both equivalent theories are incomplete.

7.1. The failure of the counts. A question that arises naturally is to characterize graphs that result in a stiff design for some configuration of circles. A quick conjecture may take the following form.

PROPOSAL 1. *If a graph G with $|V| \geq 3$ satisfies $|E| = 3|V| - 6$ and all subgraphs G' of G with $|V'| \geq 3$ satisfy $|E'| = 3|V'| - 6$, then the m-circle design is stiff.*

This is the immediate generalization of a theorem of Laman [15] characterizing edge-minimal first-order rigid bar-and-joint frameworks in the plane. This proposal does not hold in the theory of bar-and-joint frameworks, as is evident in Figure 7.1. This framework satisfies the conditions of the proposal but is obviously flexible with a rotation of one piece about the dotted line. If we construct an m-circle design on the same graph, then the resulting design is also not uniquely determined, as is evident in Figure 7.2.

7.2. Projective transformations. Since inversions are angle preserving transformations, if $\mathbf{m}$ is an m-circle configuration and we apply any inversive transformation T, then $T(\mathbf{m})$ gives an isomorphic constraint matrix for any graph. The stiffness, independence, etc. of any design $(G, \mathbf{m})$ is invariant under inversion.

Similarly, it is clear that if $\mathbf{p}$ is a spatial configuration and we apply any congruence map T, then $T(\mathbf{p})$ gives an isomorphic set of distance constraints, and for any graph the rank of the rigidity matrix is unchanged. The first-order rigidity, independence, etc. of any framework $(G, \mathbf{p})$ is invariant under congruence.

Now it is not true that an inversion in the plane induces a congruence of the 3-space paraboloid model. However, there is a family of transformations that includes both the spatial versions of inversions and the congruences of Euclidean 3-space. These are the projective transformations of 3-space.

It is well known that these projective transformations leave the rank of the corresponding rigidity matrices unchanged for any graph G and any configuration $\mathbf{m}$ where the points remain finite. (For projective points at infinity, there is a full projective form of the theory, including a projective rigidity matrix, which incorporates such constraints involving such points [3, 26, 27].)

From our correspondence, it follows that these projective transformations also leave the rank of the constraint matrix unchanged, provided that none of the points of the configuration move onto the paraboloid, where both the constraint matrix and the isomorphism are undefined. In fact, the inversive maps are precisely the projective transformations which preserve the paraboloid.

We close by stating (without proof) this conclusion.

THEOREM 7.1 (see [20]). *Given an m-circle design $(G, \mathbf{m})$ and a projective transformation T of 3-space such that $T(\mathbf{m})$ is another configuration of real circles, the constraint matrices $\mathcal{C}(G, \mathbf{m})$ and $\mathcal{C}(G, T(\mathbf{m}))$ have isomorphic row spaces.*

8. Extensions to other dimensions. Up to this point, we have studied m-circle designs for the plane and their connection to bar-and-joint frameworks in 3-space. However, the basic problems of CAD lie in three dimensions, and frameworks have been studied in all dimensions. It is natural to ask whether our results extend immediately to m-sphere designs with spheres in 3-space constrained by angles of intersection and bar-and-joint frameworks in 4-space. They do.

All the correspondences extend to designs of (hyper)spheres in n dimensions and bar-and-joint frameworks in $n + 1$ dimensions. We will present only this extension explicitly for m-sphere designs and frameworks in 4-space, but the reader will easily see how the general extension works.

As an aside, we note that there is also a correspondence between the first-order theory of plane bar-and-joint frameworks and m-interval designs along the line. While we know of no direct use for these m-interval designs, such an analysis is useful anytime an m-circle design or an m-sphere design contains a substantial piece which lies in a linear family, as this piece will behave as an m-interval design. See section 9.

For spheres in 3-space, we choose a similar normalization to that for circles and represent the sphere with equation

$$x^2 + y^2 + z^2 - 2bx - 2cy - 2dz + e = 0$$

by the four-tuple (b, c, d, e), where the orthogonal projection of (b, c, d, e) onto the xyz-space yields the center of the sphere, (b, c, d). The radius of the sphere is $b^2 + c^2 + d^2 - e$. The cosine of the angle of intersection between two intersecting spheres (b_1, c_1, d_1, e_1) and (b_2, c_2, d_2, e_2) is

$$\mathcal{K}_{1,2} = \frac{2b_1b_2 + 2c_1c_2 + 2d_1d_2 - e_1 - e_2}{2\sqrt{(b_1^2 + c_1^2 + d_1^2 - e_1)(b_2^2 + c_2^2 + d_2^2 - e_2)}}.$$

From this equation, one has the constraint equation and can easily derive the $|E| \times 4|V|$ constraint matrix $\mathcal{C}(G, \mathbf{m})$ for a design $(G, \mathbf{m})$, $\mathbf{m} \in \mathbb{R}^{4|V|}$.

The first-order theory of bar-and-joint frameworks in 4-space gives the analogous $|E| \times 4|V|$ rigidity matrix $R(G, \mathbf{m})$ for the configuration.

To translate between the first-order theory of bar-and-joint frameworks in 4-space and the m-spheres in 3-space, we use the following transformation:

$$
\mathbf{T_m} = \begin{bmatrix} \mathbf{T}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{T}_2 & \cdots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{T}_{|V|} \end{bmatrix},
$$

where

$$
\mathbf{T}_i = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & -b_i \\ 0 & -\frac{1}{2} & 0 & -c_i \\ 0 & 0 & -\frac{1}{2} & -d_i \\ -b_i & -c_i & -d_i & -2e_i \end{bmatrix}.
$$

As before, $\mathcal{C}(G, \mathbf{m}) \times \mathbf{T_m} = R(G, \mathbf{m})$ for all graphs and all configurations with no points on the paraboloid.

At this point, we hope it is obvious how the results continue to generalize into 3-space and also generalize into higher dimensions.

Remark. It is surprising to find a geometric structure in 3-space whose constraints geometrically model the problems of rigidity in 4-space. Previous work [32] provided a structure from bivariate splines in CAGD which was analogous to the generic properties of rigidity in 4-space, but this structure is known to not be even generically equivalent [32]. m-sphere designs in 3-space provide a full 3-space embodiment of the geometric and combinatorial first-order theory of frameworks in 4-space.

The limited studies of first-order rigidity in 4-space suggest major complexity that does not arise simply from the counting (matroidal) level [32]. This complexity is a warning that general geometric constraints in 3-space are much harder than the simple theory of distances in plane frameworks.

9. Circles of fixed radius. Within CAD programming, it is possible that some or all of the circles have a fixed radius. The theory presented offers a simple extension for this case, independent of whether the fixed radii are distinct numbers or all the same size.

9.1. A fixed radius. The radius r of a circle (b, c, d) is given by the equation

$$
r^2 = b^2 + c^2 - d.
$$

If r is constant, the usual process for finding the Jacobian turns this into the homogeneous linear equation,

$$
(2b)b' + (2c)c' - d' = 0.
$$

This gives a row for our constraint matrix which has only nonzero entries $[2b, 2c, -1]$ under the single circle.

If we partition the circles into two classes, indexed by V for variable radii circles and U for fixed radii circles, we simply add such a row for each circle of fixed radius
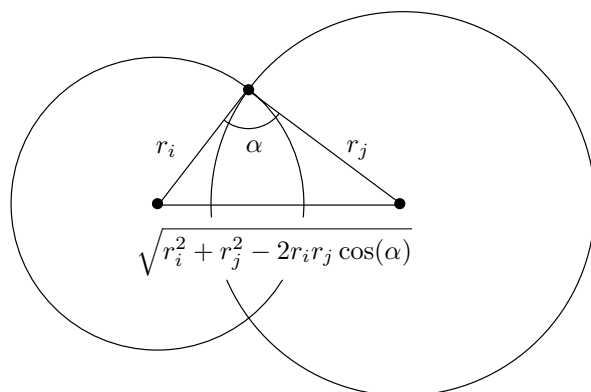
FIG. 9.1. *If we have two circles of fixed radii with a fixed angle of intersection, then the distance between the centers is also fixed.*

to create the *extended constraint matrix* $\mathcal{C}((V, U; E), \mathbf{m})$. While we could analyze this directly, it will be simpler to translate immediately to the corresponding matrix for first-order rigidity: $\mathcal{C}((V, U; E), \mathbf{m}) \times \mathbf{T_m} = R((V, U; E), \mathbf{m})$.

If we examine the transfer multiplication for the added rows for circles in U, we find

$$[2b\ 2c\ -1] \times \begin{bmatrix} -\frac{1}{2} & 0 & -b \\ 0 & -\frac{1}{2} & -c \\ -b & -c & -2d \end{bmatrix} = [b - b\ c - c\ -b^2 - c^2 + d] = [0\ 0\ -r^2].$$

We have an extended rigidity matrix that forces the third coordinates of each of the points in U to have derivative 0 (assuming $r \neq 0$ as we have throughout the entire translation process). Alternatively, we can use these rows to do a row reduction which makes all other entries in the third columns into 0 and sets off a set of $|U|$ independent rows at the bottom.

9.2. All circles of fixed radius. If $V = \emptyset$ and all radii are fixed, the extended rigidity matrix is now the rigidity matrix for a plane framework with vertices at the centers of the circles, plus an added, spread copy of the identity at the bottom. The angle constraints on the circles are isomorphic, at first-order, to distances constraints between the centers.

This equivalence is also evident from elementary geometry. If we have two circles with constant radii r_i and r_j, respectively, then the angle constraint α actually does fix the distance between the centers, and this distance is given by the law of cosines $\sqrt{r_i^2 + r_j^2 - 2r_i r_j \cos \alpha}$ (Figure 9.1). (Note that this is even true when the circles are nonintersecting and the constraint is an inversive distance. The only real limitation in this is that the circles have nonzero, though possibly imaginary, radii.)

As mentioned in section 7.1, this theory of plane points, with distance constraints or, equivalently, plane circles with fixed radii and angle constraints, has a combinatorial theory represented by the counts for independence [15]:

$$|E'| \leq 2|U'| - 3 \text{ for all nonempty subsets } U' \subseteq U.$$

Similarly for all n, if we work with hyperspheres in n-space with angle constraints and all radii fixed, we find the theory is isomorphic to the theory of points in n-space with distance constraints.
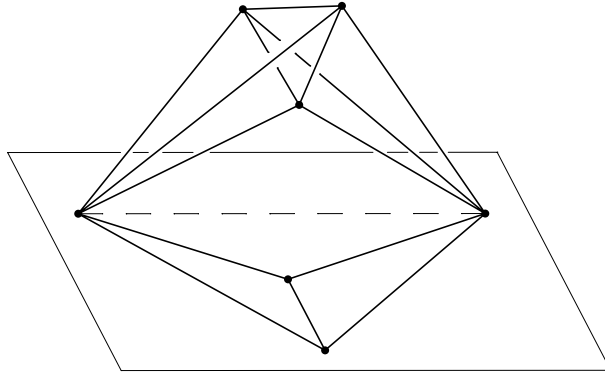
FIG. 9.2. *Although the mixed framework satisfies all the necessary counting conditions for mixed structures, the framework is not rigid: there is a rotation of one half of the framework about the dotted line.*

9.3. Some circles of fixed radius. A more interesting middle situation is where we have some circles of fixed radii and others of variable radii. Equivalently, we may be looking to see which subsets of angles and radii are independent constraints in a design, as new constraints of either type are considered for addition to a currently independent design.

Assume that $|V| \geq 3$ and $|U| \geq 3$. With this assumption on $|U|$, the only trivial motions are generated by the translations parallel to the x- and y-axes and rotation about the z-axis, which occurred for plane rigidity. (The reader can verify this, even in the original circle constraint representation with the initial list of generators for the trivial motions and the initial representation of a fixed radius constraint.) This leaves the following obvious counting condition for independence:

$$|E'| \leq 3|V'| + 2|U'| - 3 \quad \text{for all } |U'| \geq 3.$$

If $|U| < 3$, we have other spaces of trivial motions, as follows:
1. for $|U| = 2$ and $|V| \geq 1$, the space of trivial motions has dimension 4, adding the 3-space rotations about the line through the two points in U;
2. for $|U| = 1$ and $|V| \geq 2$, the space of trivial motions has dimension 5, adding the 2-space of spatial rotations fixing this point to the trivial motions of the plane;
3. for $|U| = 0$ and $|V| \geq 3$, the space of trivial motions has dimension 6.

These observations can be pulled together into necessary counting conditions for independence. However, provided the design is large enough to contain structures such as the two bananas of Figure 7.1 or the adapted mixed version of Figure 9.2, these conditions will not be sufficient for independence.

9.4. Lines among circles of fixed radius. All of this analysis for fixed radii works within our simplifying assumption that our m-circles did not include lines. Of course, with no fixed radii, or only one, we could use inversion (fixing the radius) to pull the lines into circles and continue the analysis of this equivalent configuration containing no lines.

However, with several fixed radii, we are restricted in our transformations, and lines are not incorporated into the simple theory. Intuitively, if we fix the "radii of lines" as circles of infinite radius, then we would restrict them to remain lines. (The

transformations should be reversible, and no fixed finite radius can become infinite.) By an elementary observation, an angle constraint between a circle of fixed radius and a line fixed to remain a line, still fixes the distance from the center of the circle to the line.

A detailed analysis here would again require a fully projective presentation, using homogeneous coordinates for all points in the model (or, equivalently, for lines and circles, including the effects of fixed radii). It can be done if there are specific situations where this would be significant.

9.5. Fixed radius circles in the spherical model. If we work with circles and angle constraints on the sphere, the condition for a fixed radius changes. Recall that a circle is represented by the point at the tip of a cone tangent to the sphere at the circle: $t = (x, y, z)$. To hold the radius of such a circle fixed, we just fix the distance (squared) from this point to the center of the sphere (the origin): $x^2 + y^2 + z^2 = d^2$. In the Jacobian, this gives the row whose only nonzero entries are (x, y, z). This is equivalent, for any analysis of independence or dependence of constraints, to adding the center of the sphere as a vertex and a bar from this center to the circle point, creating an overall 3-space framework for these radial constraints and the angle constraints. This equivalence holds both combinatorially and at the specific geometric level of possible special positions in which a generically independent set of constraints drops in rank and becomes dependent.

If we constrain the radius of each of the circles, then we have the center of the sphere as a vertex joined to all the other vertices. By the cone theorem for frameworks [28, 32], this is equivalent to the constraint matrices for the framework created by projecting from the center onto a projection plane tangent to the sphere at $z = 1$. (This is a general projection, so points below the equatorial plane are joined to the center and the line extended to intersect the projection plane.) It is also equivalent to the spherical framework in which each of the circle vertices is pulled onto the initial sphere (at the center of the original circle), and one studies the framework constrained to remain on the sphere.

It is worth recalling that the central projection to a plane framework is different from the stereographic projection into a plane circle configuration from the top of the sphere. The plane framework here is distinct from the framework in which the plane radii were fixed. In general, these two frameworks for fixed radii in the plane and fixed radii on the sphere are not even projectively equivalent. They do have the same graph and will, generically, have the same independence structure. However, for special positions arising from the geometric placement of the plane vertices, they may have distinct behavior.

10. Concluding remarks. We have analyzed correspondences among circles and lines in the inversive plane, circles on the sphere, and points in Euclidean (and hyperbolic) space. An identical pattern happens for other dimensions. For example, if we study the angle constraints between intersecting (and nonintersecting) spheres in inversive 3-space, these are isomorphic to angle constraints between corresponding hyperplanes in hyperbolic 4-space and the $\mathcal{K}$ constraints among ideal points in hyperbolic 4-space. At first order, they are also identical to the distance constraints between points in Euclidean 4-space, as we noted in section 8.

There are geometric connections between these interconnected problems of circles in the plane or the sphere and Andreev's theorem and its extensions [1]. Via the correspondence offered here and the related correspondences in [21], these results of Andreev are also connected to the rigidity theorems for convex polyhedra of Cauchy

and Alexandrov (see [19, 30]). We will explore these connections more extensively in a forthcoming paper.

By giving a correspondence between angle constraints in the inversive plane and distance constraints in the Euclidean space, we raised the question of a polynomial time algorithm for the generic rank of a configuration of circles and angles. The corresponding unsolved problem for 3-space has been studied, and conjectured about, for over a century. At least one other plane geometric problem, that of bivariate C_2^1-splines, is also conjectured to be isomorphic at a generic (but not a geometric) level for the rank of a corresponding matrix on a given graph [32]. The study of each of these variants has contributed to our store of shared techniques and results, but we need new approaches to solve the shared problem.

A natural question to ask is whether the situation with m-circles can contribute any additional insights. In [21] we describe the equivalence of first-order rigidity in all the Cayley–Klein geometries extracted from the underlying projective geometry, including the hyperbolic, spherical, and Euclidean spaces [6]. This isomorphism suggests that we will not easily find new combinatorial results by switching so transparently among these equivalent theories.

There remain other variants of these problems of plane objects such as points, lines, and circles with geometric constraints in CAD. Many are unsolved, and some are simply unstudied. Consider including points which are assigned to lie on one or several circles in an m-circle design. In its most general form, such an incidence pattern would include the projective configurations of lines and incident points. After all, lines are simply inversive circles which all share a common point (chosen to be the inversive point at infinity). Without some additional restrictions, this problem with circles should be at least as hard as the specific problem of incident lines and points in the plane, which we have previously conjectured to have no polynomial time algorithm [32].

However, if we insist that all circles intersect at one specific point, with fixed angles at that point, then we have a problem which has been solved. This is inversively equivalent to the problem of lines with fixed angles—or, equivalently, parallel drawings of configurations of lines—and has a polynomial time algorithm related to the counting algorithms for the generic rigidity for plane frameworks [22, 31]. Moreover, the analogous problem for parallel drawings of planes in 3-space with fixed angles also has a polynomial time algorithm [31].

If we drop the condition that all angles at the common point of intersection are fixed, we return, once more, to an unsolved (and difficult) problem. In the plane, we have studied lines, incidences, and angles with additional simplifying restrictions that all lines are "short" (have no more than two assigned incident points). These incidence constraints can then be extended by additional selected angle constraints among the points and lines. Even this very special case is hard and is conjectured not to have a polynomial time algorithm [2].

All of these interconnected problems, many unsolved, confirm the complexity of various sets of constraints in plane and spatial CAD. The specific case of points and distance constraints in the plane, plane first-order rigidity, stands out as an exceptional case in which we do have a polynomial time algorithm. This is not typical of constraints in CAD, and strategies based on the assumption of polynomial time algorithms for even generic rigidity or independence of constraints are limited in their applications [10, 11]. The need for fast symbolic algorithms does, in effect, restrict the patterns of constraints that are handled well in CAD programs, before resorting to more brute force (and more unstable) numerical analysis of the constraints [16, 17].

The study of various sets of constraints, even in plane CAD, continues to generate rich connections backward into classical geometry in all its forms, new connections among these classical problems, and new insights. When we started this investigation of circles and angles, we had no expectation that it would lead to hyperbolic 3-space and correspondences to Euclidean space. We look forward, with anticipation, to the next piece of the puzzle and the connections it will bring forward for our geometric play.

REFERENCES

[1] E. M. ANDREEV, *Convex polyhedra in Lobačevskiĭ spaces,* Mat. Sb. (N.S.), 81 1970, pp. 445–478.

[2] M. BOUSFIELD, K. CALDWELL, L. DUONG, D. MOSCOVITZ, AND W. WHITELEY, *Constraining Plane Configurations in CAD: Angles,* preprint, York University, North York, ON, Canada, 1996.

[3] H. CRAPO AND W. WHITELEY, *Statics of frameworks and motions of panel structures: A projective geometric introduction,* Structural Topology, 6 (1982), pp. 43–82.

[4] R. CONNELLY AND W. WHITELEY, *Second-order rigidity and prestress stability for tensegrity frameworks*, SIAM J. Discrete Math., 9 (1996), pp. 453–491.

[5] R. CONNELLY, *On generic global rigidity,* in Applied Geometry and Discrete Mathematics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 4, AMS, Providence, RI, 1991, pp. 147–155.

[6] H. S. M. COXETER, *An Introduction to Non-Euclidean Geometry,* 6th ed., Mathematical Association of America, Washington, D.C., 1998.

[7] J. GRAVER, B. SERVATIUS, AND H. SERVATIUS, *Combinatorial Rigidity*, AMS, Providence, RI, 1994.

[8] B. HENDRICKSON, *Conditions for unique graph realizations*, SIAM J. Comput., 21 (1992), pp. 65–84.

[9] L. HENNEBERG, *Die graphische Statik der Starren Systeme*, Leipzig 1911, Johnson Reprint, 1968.

[10] C. HOFFMAN, A. LOMONOSOV, AND M. SITHARAM, *Decomposition plans for geometric constraint systems, part* I: *Performance measures for CAD,* J. Symbolic Comput., 31 (2001), pp. 367–408.

[11] C. HOFFMAN, A. LOMONOSOV, AND M. SITHARAM, *Decomposition plans for geometric constraint systems, part* II: *New algorithms,* J. Symbolic Comput., 31 (2001), pp. 409–428.

[12] C. HOFFMAN AND C.-S. CHIANG, *Variable-radius circles in cluster merging, part* I, CAD, 34 (2002), pp. 787–797.

[13] C. HOFFMAN AND C. S. CHIANG, *Variable-radius circles in cluster merging, part* II, CAD, 34 (2002), pp. 799–805.

[14] G. KRAMER, *Solving Geometric Constraint Systems (A Case Study in Kinematics),* MIT Press, Cambridge, MA, 1992.

[15] G. LAMAN, *On graphs and the rigidity of plane skeletal structures,* J. Engrg. Math., 4 (1970), pp. 331–340.

[16] J. C. OWEN, *Algebraic solutions for geometry from dimensional constraints,* in Proceedings of the First Symposium on Solid Modeling Foundations and CAD/CAM Applications, Austin, TX, 1991, ACM, New York, 1991, pp. 397–407.

[17] J. C. OWEN, *Constraints on simple geometry in two and three dimensions,* Internat. J. Comput. Geom. Appl., 6 (1996), pp. 421–434.

[18] D. PEDOE, *Geometry: A Comprehensive Course,* Dover, New York, 1988.

[19] A. V. POGORELOV, *Extrinsic Geometry of Convex Surfaces,* Transl. Math. Monogr. 35, AMS, Providence, RI, 1973.

[20] B. ROTH AND W. WHITELEY, *Tensegrity frameworks,* Trans. Amer. Math. Soc., 177 (1981), pp. 419–446.

[21] F. SALIOLA AND W. WHITELEY, *The First-Order Equivalence of Rigidity in Euclidean, Hyperbolic and Spherical Geometries,* preprint, York University, North York, ON, Canada.

[22] B. SERVATIUS AND W. WHITELEY, *Constraining plane configurations in computer-aided design: Combinatorics of directions and lengths,* SIAM J. Discrete Math., 12 (1999), pp. 136–153.

[23] K. SUGIHARA, *Detection of structural inconsistency in systems of equations with degrees of freedom,* J. Discrete Appl. Math., 10 (1985), pp. 297–312.

[24] T.-S. TAY, *On the generic rigidity of bar-frameworks,* Adv. in Appl. Math., 23 (1999), pp. 14–28.

[25] T.-S. TAY AND W. WHITELEY, *Generating all isostatic frameworks,* Structural Topology, 11 (1985), pp. 21–69.

[26] T.-S. TAY, N. WHITE, AND W. WHITELEY, *Skeletal rigidity of symplicial complexes* I, European J. Combin., 16 (1995), pp. 381–403.

[27] T.-S. TAY, N. WHITE, AND W. WHITELEY, *Skeletal rigidity of symplicial complexes* II, European J. Combin., 16 (1995), pp. 503–523.

[28] W. WHITELEY, *Cones, infinity and one-story buildings,* Structural Topology, 8 (1983), pp. 53–70.

[29] W. WHITELEY, *Infinitesimal rigidity of a bipartite framework,* Pacific J. Math., 110 (1984), pp. 233–255.

[30] W. WHITELEY, *Infinitesimally rigid polyhedra* I*: Statics of frameworks,* Trans. Amer. Math. Soc., 285 (1984), pp. 431–461.

[31] W. WHITELEY, *Some matroids on hypergraphs with applications to scene analysis and geometry,* Discrete Comput. Geom., 4 (1988), pp. 75–95.

[32] W. WHITELEY, *Matroids for discrete applied geometry,* in Matroid Theory, Contemp. Math. 197, J. Bonin, J. Oxley, and B. Servatius, eds., AMS, Providence, RI, 1996, pp. 171–311.

[33] W. WHITELEY, *Constraining Plane Geometric Configurations in CAD: Directions and Distances,* preprint, Department of Mathematics and Statistics, York University, North York, ON, Canada, 1999.

[34] J. WILKER, *Inversive geometry,* in The Geometric Vein: The Coxeter Festschrift, C. Davis, B. Grunbaum, and F. A. Sherk, eds., Springer, New York, 1981, pp. 379–442.