EIGENVECTORS FOR A RANDOM WALK ON A LEFT-REGULAR BAND

FRANCO SALIOLA

Abstract. We present a simple construction of the eigenvectors for the transition matrices of random walks on a class of semigroups called left-regular bands. These walks were introduced and analyzed by Brown, and they include the hyperplane chamber walks of Bidigare, Hanlon and Rockmore. This construction leads to new concise proofs of several of the known results about these walks. We also explain how tools from poset topology can be used to extract an eigenbasis for the transition matrices of the hyperplane chamber walks, and indicate the connection with a method recently described by Denham.

1. Introduction

In [3], Brown introduces and analyzes random walks on a class of semigroups called left-regular bands. These encompass several well-known random walks, including the walks on the chambers of a hyperplane arrangement introduced by Bidigare, Hanlon and Rockmore [1]. The transition matrices of these walks are diagonalizable and their eigenvalues are easy to describe (see §2).

We present here a simple construction of the eigenvectors for the transition matrices of these random walks, leading to concise proofs of several of the known results about these walks. We follow Brown's lead and consider the transition matrices as elements of the semigroup algebra. In §3, we decompose these elements into linear combinations of orthogonal idempotents by specializing a simple recursive procedure introduced in [7]. We use this decomposition to describe the eigenspaces of the transition matrices and to derive several known results. In §4, we explain how to use tools from poset topology to extract an eigenbasis for the transition matrices, and we indicate the connection with a method recently described by Denham [5].

We remark that Steinberg [9] also produced a simple proof of Brown's diagonalizability result, but his proof does not yield the eigenvectors.

2. Notation & Background

2.1. **Left-regular bands.** Let S be a finite semigroup with identity. S is said to be a *left-regular band* if there is a lattice $\mathcal{L}$ together with a surjection $\operatorname{supp} : S \to \mathcal{L}$ satisfying

$$\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$$

and

$$xy = x \quad \text{if} \quad \operatorname{supp}(y) \leq \operatorname{supp}(x)$$

for all x and y in S, where $\vee$ denotes the join operation of $\mathcal{L}$.

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An element $x \in S$ is said to be a *chamber* if $\operatorname{supp}(x) = \hat{1}$, where $\hat{1}$ is the (unique) maximal element of $\mathcal{L}$.

It follows easily from this definition that the elements of a left-regular band satisfy the identities $x^2 = x$ and $xyx = xy$. (In fact, this is a characterization of left-regular bands [3, Appendix B].)

2.2. **Semigroup of faces of a hyperplane arrangement.** Hyperplane arrangements provide one source of examples of left-regular bands. We recall here the relevant notions and refer the reader to [3, Appendix A] for details.

Let $\mathcal{A}$ denote a central hyperplane arrangement in $V = \mathbb{R}^n$. The hyperplanes in $\mathcal{A}$ partition V into subsets called the *faces* of the arrangement. There is a natural semigroup structure on the set of faces $\mathcal{F}$ defined as follows. The product of two faces x and y is the face xy uniquely defined by the properties that, for each hyperplane $H \in \mathcal{A}$, the points in xy lie: on the same side of H as x if $x \not\subseteq H$; on the same side of H as y if $x \subseteq H$, but $y \not\subseteq H$; and inside H if $x, y \subseteq H$. This product admits an alternative geometric description: xy is the unique face one first enters (possibly x itself) when following a straight line from a point in x toward a point in y.

Let $\mathcal{L}$ denote the intersection lattice of $\mathcal{A}$; that is, the set of subspaces of V that can be expressed as an intersection of hyperplanes in $\mathcal{A}$. We order $\mathcal{L}$ by inclusion, and remark that some authors choose to order $\mathcal{L}$ by reverse-inclusion. The map $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$ that sends a face x to the smallest subspace in $\mathcal{L}$ containing x satisfies the conditions of §2.1, making $\mathcal{F}$ a left-regular band. The chambers of a hyperplane arrangement are precisely the faces of maximal dimension.

2.3. **Random walks on left-regular bands.** Let S be a left-regular band, $\mathcal{C}$ the set of chambers in S and let $\{p_x\}_{x \in S}$ be a probability distribution on S. A step in the random walk moves from a chamber c to the chamber xc with probability p_x (note that the chambers form a two-sided ideal of S, so xc is a chamber since c is a chamber). The transition matrix of this walk is the matrix with rows and columns indexed by $\mathcal{C}$, and with (c, d)-entry given by

$$T_{c,d} = \sum_{xc=d} p_x. \tag{1}$$

By identifying $\{p_x\}_{x \in S}$ with the following element of the semigroup algebra $\mathbb{R}S$,

$$p = \sum_{x \in S} p_x x$$

the transformation matrix T becomes the matrix (acting on row vectors) of the linear transformation $a \mapsto pa$ restricted to the vector space $\mathbb{R}\mathcal{C}$ with basis $\mathcal{C}$.

3. Eigenvalues and Eigenspaces

In light of the above identification, to study the transition matrix we need only study the linear transition $a \mapsto pa$. We will derive an expression for p of the form

$$p = \sum_{X \in \mathcal{L}} \lambda_X e_X \tag{2}$$

where the λ_X are the eigenvalues of p and the e_X are orthogonal idempotents in $\mathbb{R}S$ (an *idempotent* is an element a satisfying $a^2 = a$; two idempotents a and b are *orthogonal* if $ab = 0 = ba$). Our starting point is the following result that allows us to construct orthogonal idempotents in $\mathbb{R}S$ that sum to 1.

Remark 1. Brown proved that the transition matrices are diagonalizable by showing that the subalgebra generated by p is semisimple. A byproduct of his proof was an expression for p of the form (2). We begin with a simple argument establishing (2), from which we immediately deduce the semisimplicity and the diagonalizability.

Theorem 2 ([7]). *For each $X \in \mathcal{L}$, let $S_X = \{x \in S : \operatorname{supp}(x) = X\}$ and fix an element $u_X = \sum_{x \in S_X} u_x x \in \mathbb{R}S_X$ with $\sum_x u_x = 1$. Define elements $e_X \in \mathbb{R}S$, one for each $X \in \mathcal{L}$, recursively using the formula*

$$e_X = u_X - \sum_{Y > X} u_X e_Y. \tag{3}$$

Then $\{e_X\}_{X \in \mathcal{L}}$ is a complete system of primitive orthogonal idempotents in $\mathbb{R}S$. (In particular, $e_X^2 = e_X$ for $X \in \mathcal{L}$, $e_X e_Y = 0$ if $X \neq Y$, and $\sum_{X \in \mathcal{L}} e_X = 1$.)

These idempotents also satisfy the following remarkable property.

Lemma 3 ([7]). *Let $X \in \mathcal{L}$ and $y \in S$. If $\operatorname{supp}(y) \not\leq X$, then $ye_X = 0$.*

(We do not include proofs of these results here, but remark that they can be proved by first establishing Lemma 3, then using it to prove Theorem 2.)

The decomposition for p given in (2) results from a specialization of the u_X in Theorem 2. The following arguments do not require that $\sum_{x \in S} p_x = 1$, so we temporarily drop this assumption and write $\lambda_V = \sum_{x \in S} p_x$.

It follows from (1) that T is a nonnegative matrix since $p_x \geq 0$ for all x in S. The sum of the entries in the row of T indexed by the chamber c is

$$\sum_{d \in \mathcal{C}} T_{c,d} = \sum_{d \in \mathcal{C}} \sum_{\substack{x \in S \\ xc = d}} p_x = \sum_{x \in S} p_x = \lambda_V,$$

which is independent of the chamber c. Hence, the row sums of T are all equal to λ_V, from which it follows that λ_V is both an eigenvalue of T and its spectral radius [6, Lemma 8.1.21]. By a generalization of the Perron-Frobenius Theorem to nonnegative matrices [6, Lemma 8.3.1], there is a nonnegative and nonzero row vector $\vec{u}$ such that $\vec{u}\,T = \lambda_V T$. Viewing $\vec{u}$ as the element $u := \sum_{c \in \mathcal{C}} u_c c \in \mathbb{R}\mathcal{C}$, we have $pu = \lambda_V u$ and $\sum_c u_c \neq 0$. In particular, we renormalise so that $\sum_c u_c = 1$.

Fix $X \in \mathcal{L}$ and apply the above argument to the element

$$p_X = \sum_{\substack{x \in S \\ \operatorname{supp}(x) \leq X}} p_x x$$

acting by left-multiplication on $\mathbb{R}S_X$, where $S_X = \{x \in S : \operatorname{supp}(x) = X\}$. Then, for each $X \in \mathcal{L}$, there exists an element $u_X = \sum_{x \in S_X} u_x x \in \mathbb{R}S_X$ satisfying

$$p_X u_X = \lambda_X u_X \quad \text{and} \quad \sum_{x \in S_X} u_x = 1, \quad \text{where} \quad \lambda_X = \sum_{\operatorname{supp}(x) \leq X} p_x. \tag{4}$$

With these u_X as input, define elements $e_X \in \mathbb{R}S$ recursively using (3). By Theorem 2, the e_X are orthogonal idempotents that sum to the identity in $\mathbb{R}S$.

Theorem 4. *Let $p = \sum_{x \in S} p_x x$ with $p_x \geq 0$ for all $x \in S$. Then*

$$p = \sum_{X \in \mathcal{L}} \lambda_X e_X,$$

where $\{e_X\}_{X \in \mathcal{L}}$ are the idempotents defined in the preceding paragraph.

Proof. Note that $u_X^2 = u_X$: since $yx = y$ for $x, y \in S_X$ and $\sum_x u_x = 1$, we have

$$u_X^2 = \sum_{y,x \in S_X} u_y u_x yx = \sum_{y,x \in S_X} u_y u_x y = \left(\sum_{x \in S_X} u_x \right) u_X = u_X.$$

By Lemma 3,

$$pe_X = \left(\sum_{y \in S} p_y y \right) e_X = \left(\sum_{\operatorname{supp}(y) \leq X} p_y y \right) e_X = p_X e_X.$$

Since u_X is an idempotent, we have $e_X = u_X e_X$. Combining this with (4) yields

$$pe_X = p_X e_X = p_X u_X e_X = \lambda_X u_X e_X = \lambda_X e_X.$$

Finally, since $\sum_{X \in \mathcal{L}} e_X = 1$,

$$p = \sum_{X \in \mathcal{L}} pe_X = \sum_{X \in \mathcal{L}} \lambda_X e_X. \quad \square$$

From this we recover Brown's results that the subalgebra of $\mathbb{R}S$ generated by p is split semisimple [3]. Furthermore, it allows us to easily describe the eigenvalues and eigenspaces of the transition matrices of the random walks.

Corollary 5. *Keep the notation of Theorem 4. There is a direct sum decomposition of vector spaces $\mathbb{R}\mathcal{C} \cong \bigoplus_{X \in \mathcal{L}} e_X \mathbb{R}\mathcal{C}$ and left-multiplication by p on the subspace $e_X \mathbb{R}\mathcal{C}$ is scalar multiplication by λ_X. Thus, the λ-eigenspace of p is $\bigoplus_{\lambda_X = \lambda} e_X \mathbb{R}\mathcal{C}$.*

Proof. Since the idempotents e_X are orthogonal and sum to 1, the vector space $\mathbb{R}\mathcal{C}$ decomposes into an internal direct sum of the subspaces $e_X \mathbb{R}\mathcal{C}$. Since $p = \sum_{X \in \mathcal{L}} \lambda_X e_X$, the action of p on $e_X \mathbb{R}\mathcal{C}$ is scalar multiplication by λ_X. $\square$

We also obtain a combinatorial criterion for the existence of a unique stationary distribution for the random walk. A distribution $\{p_x\}_{x \in S}$ is said to be *separating* if for each $H \in \mathcal{L}$ with $H \lessdot \hat{1}$, there exists $x \in S$ with $\operatorname{supp}(x) \not\leq H$ and $p_x > 0$.

Corollary 6 ([4])**.** *T has a unique stationary distribution if and only if the probability distribution $\{p_x\}_{x \in S}$ is separating.*

Proof. If T has a unique stationary distribution, then the multiplicity of $\lambda_{\hat{1}}$ is 1. Hence, $\lambda_{\hat{1}} > \lambda_X$ for all $X \neq \hat{1}$. Since $\lambda_{\hat{1}} > \lambda_X$, there exists $x \in S$ with $\operatorname{supp}(x) \not\leq X$ and $p_x > 0$. If p is separating, then $\lambda_{\hat{1}} > \lambda_H$ for all $H \in \mathcal{L}$ with $H \lessdot \hat{1}$, and so $\lambda_{\hat{1}} > \lambda_X$ for all $X \neq \hat{1}$. Hence, the multiplicity of $\lambda_{\hat{1}}$ is $\dim(e_{\hat{1}} \mathbb{R}\mathcal{C}) = \dim(\mathbb{R} e_{\hat{1}}) = 1$. $\square$

We also obtain the eigenvalue multiplicities for the hyperplane chamber walks.

Corollary 7 ([1], [4])**.** *Let T be the transition matrix for the random walk on the chambers of a hyperplane arrangement driven by the distribution $\{p_x\}_{x \in \mathcal{F}}$. The eigenvalues of T are $\{\lambda_X\}_{X \in \mathcal{L}}$, and the multiplicity of λ_X is $\sum_{\lambda_Y = \lambda_X} |\mu(X, V)|$, where μ is the Möbius function of $\mathcal{L}$.*

Proof. Most of this follows from the previous results; the only thing needing proof is the description of the multiplicities. From the decomposition $\mathbb{R}\mathcal{C} = \oplus_{X \in \mathcal{L}} e_X \mathbb{R}\mathcal{C}$, one has $\dim(\mathbb{R}\mathcal{C}) = \sum_X \dim(e_X \mathbb{R}\mathcal{C})$. The left side counts the number of chambers of $\mathcal{A}$. Comparing this with Zaslavzky's formula for the number of chambers of an arrangement [11], we conclude that $\dim(e_X \mathbb{R}\mathcal{C}) = |\mu(X, V)|$. $\square$

4. Eigenbases

In this section we indicate how to use tools from poset topology [10] to extract an eigenbasis for the transition matrices of the hyperplane chamber walks.

A *saturated chain* in $\mathcal{L}$ is a collection of nested subspaces $X_0 \subset X_1 \subset \cdots \subset X_r$ in $\mathcal{L}$ with $\dim(X_i) = \dim(X_{i-1}) + 1$ for every $1 \leq i \leq r$. Let Q denote the set of saturated chains of $\mathcal{L}$ and denote by $\mathbb{R}Q$ the $\mathbb{R}$-vector space with basis the elements of Q. Then $\mathbb{R}Q$ is an algebra; the product of two saturated chains $X_0 \subset \cdots \subset X_r$ and $Y_0 \subset \cdots \subset Y_s$ is $X_0 \subset \cdots \subset X_r = Y_0 \subset \cdots \subset Y_s$ if $X_r = Y_0$ and 0 otherwise. (In the standard terminology of representation theory of algebras, Q describes the set of paths of the *quiver* of $\mathbb{R}\mathcal{F}$ and $\mathbb{R}Q$ is the *path algebra* of the quiver.)

In [8], an algebra surjection $\varphi : \mathbb{R}Q \to \mathbb{R}\mathcal{F}$ was constructed. This map can be defined so that the image of the vertices of Q (viewed as chains in Q of length 0) is any complete system of primitive orthogonal idempotents. In particular, we can define φ so that $\varphi(X) = e_X$ for all $X \in \mathcal{L}$, where e_X is given by Theorem 4.

In order to define the images of the arrows, fix an *orientation* ϵ_X on each subspace $X \in \mathcal{L}$: thus, ϵ_X is a function that maps an ordered basis of X to $+1$ or -1 in such a way that $\epsilon_X(\vec{v}_1, \ldots, \vec{v}_d) = \epsilon_X(\vec{u}_1, \ldots, \vec{u}_d)$ if and only if the change of basis matrix mapping $\vec{v}_1, \ldots, \vec{v}_d$ to $\vec{u}_1, \ldots, \vec{u}_d$ has positive determinant. Then define numbers $[y : x] \in \{\pm 1\}$ for faces x and y with $\operatorname{supp}(y) \lessdot \operatorname{supp}(x)$ by

$$[y : x] = \epsilon_{\operatorname{supp}(x)}(\vec{y}_1, \ldots, \vec{y}_d, \vec{x}_1), \tag{5}$$

where $\vec{y}_1, \ldots, \vec{y}_d$ is a positively-oriented basis of $\operatorname{supp}(y)$ and $\vec{x}_1$ is a vector in x. Then the image of a saturated chain $Y \subset X$ in Q of length 2 is defined to be

$$\varphi(Y \subset X) = e_Y \Big([y : x]x + [y : x']x' \Big) e_X,$$

where y is any face with $\operatorname{supp}(y) = Y$ and x, x' are two faces on opposite sides of Y with $\operatorname{supp}(x) = \operatorname{supp}(x') = X$. Then φ extends uniquely to a surjection of algebras, and the kernel of φ is generated as an $\mathbb{R}Q$-ideal by the sum of all the chains of length two in Q. It follows from Lemma 3 and the left-regular band identities (§2.1) that the image under φ of the saturated chain $X_0 \subset X_1 \subset \cdots \subset X_r$ is of the form

$$\pm u_{X_0} \Big(x_1 - x_1' \Big) \Big(x_2 - x_2' \Big) \cdots \Big(x_r - x_r' \Big) e_{X_r},$$

where x_i, x_i' are two faces on opposite sides of X_{i-1} with $\operatorname{supp}(x_i) = \operatorname{supp}(x_i') = X_i$.

Remark 8. Recent work of Denham [5] gives a quite different approach to describing the eigenvectors for the hyperplane chamber walks. He defines a map which is essentially the restriction of φ to the subspace spanned by the saturated chains terminating with V. Denham's map has its origins in the study of the cohomology of the complement of the complexified arrangement, whereas φ arises naturally in the study of the representation theory of the semigroup algebra $\mathbb{R}\mathcal{F}$.

The image of the subspace of $\mathbb{R}Q$ spanned by all the maximal chains in the interval $[X, Y] = \{U \in \mathcal{L} : X \subseteq U \subseteq Y\}$ is the subspace $e_X \mathbb{R}\mathcal{F} e_Y$. If $Y = V$, then this image is $e_X \mathbb{R}\mathcal{F} e_V = e_X \mathbb{R}\mathcal{C}$, which by Corollary 5 corresponds to an eigenspace of the transition matrix. In particular, we get an eigenbasis by choosing appropriate sets of maximal chains in the intervals $[X, V]$. We can make use of tools from poset topology to identify suitable choices, as we now describe.

The subspace $e_X \mathbb{R}\mathcal{F} e_Y$ is isomorphic to the poset cohomology space $H^*(X,Y)$ of the interval $[X,Y]$; indeed, from the description of $\ker(\varphi)$, the subspace is isomorphic to the vector space whose basis consists of the maximal chains in $[X,Y]$ modulo the so-called "cohomology relations" (see [8] for the details).

There are various tools and techniques from poset topology that determine bases for these cohomology spaces. One of these is an *EL-labelling* of the lattice $\mathcal{L}$ [10]. Such a labelling identifies a set of distinguished maximal chains for every interval $[X,Y]$ in $\mathcal{L}$, and these chains form a basis of $H^*(X,Y)$. Moreover, it also affords a "straightening" algorithm that expresses an arbitrary maximal chain in $[X,Y]$ as a linear combination of the distinguished chains.

The map φ will send these distinguished chains to an eigenbasis of $\mathbb{R}\mathcal{F}$, and the distinguished maximal chains in $[X,V]$ to an eigenbasis of $e_X\mathbb{R}\mathcal{C}$. Thus, we obtain an eigenbasis for the transition matrices of the hyperplane chamber walks. Various EL-labellings of the lattices $\mathcal{L}$ can be found in the literature. For instance, [2] produces one in which the distinguished chains are labelled by non-broken circuits. This and other examples can be found in [10].

The quiver Q of a left-regular band S can also be described by certain chains (not necessarily saturated) in the corresponding lattice $\mathcal{L}$ [7]. However, the kernel of the surjection $\mathbb{R}Q \to \mathbb{R}S$ is not well understood at this level of generality.

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Département de Mathématiques, Université du Québec à Montréal, C.P. 8888 Succ. Centre-Ville, Montréal, Québec, H3C 3P8 Canada

E-mail address: `saliola.franco@uqam.ca`