THE FACE SEMIGROUP ALGEBRA OF A

HYPERPLANE ARRANGEMENT

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The first part of this thesis studies the face semigroup algebra of a hyperplane arrangement. The quiver with relations of the algebra is computed and the algebra is shown to be a Koszul algebra. Other algebraic structure is determined: a construction for a complete system of primitive orthogonal idempotents; projective indecomposable modules; Cartan invariants; projective resolutions of the simple modules; Hochschild (co)homology; and the Koszul dual algebra. It is shown that the algebra depends only on the intersection lattice of the hyperplane arrangement. A new cohomology construction on posets is introduced and it is shown that the face semigroup algebra is the cohomology algebra of the intersection lattice.

In the second part, attention is restricted to arrangements arising from finite reflection groups. The reflection group acts on the face semigroup algebra, and the subalgebra invariant under the group action is studied. The quiver is determined for the case of the symmetric group. Since the invariant subalgebra is anti-isomorphic to Solomon's descent algebra, the quiver of the descent algebra of the symmetric group is obtained.

The third part extends some of the results of the first part to a class of semigroups called left regular bands. In particular, a description of the quiver of the semigroup algebra is given, and it is used to compute the quiver of the face semigroup algebra of a hyperplane arrangement and of the free left regular band.

BIOGRAPHICAL SKETCH

Franco Valentino Saliola was born on 12 February 1977 in Toronto Canada to Italia Romano and Vincenzo Mario Saliola. He atteneded St Augustine's Catholic Elementary School and Regina Pacis Catholic Secondary School in Toronto. He completed a Specialized Honours Bachelor of Science in Mathematics at York University in 2000. He was admitted to the Mathematics graduate program at Cornell University in Fall 2000, received his MS in Mathematics in August 2003, and completed his PhD in August 2006 under the supervision of Dr Kenneth S Brown.

For my parents. Your support and sacrifices have made this possible.

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CHAPTER 1

THE FACE SEMIGROUP ALGEBRA OF A HYPERPLANE ARRANGEMENT

1.1 Introduction

Let $\mathcal{A}$ denote a finite collection of linear hyperplanes in $\mathbb{R}^d$. Then $\mathcal{A}$ dissects $\mathbb{R}^d$ into open subsets called *chambers*. The closures of the chambers are polyhedral cones whose relatively open faces are called the *faces* of the hyperplane arrangement $\mathcal{A}$. The set $\mathcal{F}$ of faces of $\mathcal{A}$ can be endowed with a semigroup structure. Geometrically, the product xy of faces x and y is the face entered by moving a small positive distance along a straight line from x towards y. The k-algebra spanned by the faces of $\mathcal{A}$ with this multiplication is the *face semigroup algebra* of the hyperplane arrangement $\mathcal{A}$. Here k denotes a field.

The face semigroup algebra $k\mathcal{F}$ has enjoyed recent attention due mainly to two interesting results. The first result is that a large class of seemingly unrelated Markov chains can be studied in a unified setting via the semigroup structure on the faces of a hyperplane arrangement. The Markov chains are encoded as random walks on the chambers of a hyperplane arrangement [Bidigare et al., 1999]. A step in this random walk moves from a chamber to the product of a face with the chamber according to some probability distribution on the faces of the arrangement. This identification associates the transition matrix of the Markov chain with the matrix of a linear transformation on the face semigroup algebra of the hyperplane arrangement. Questions about the Markov chain can then be answered using algebraic techniques [Brown, 2000]. For example, a combinatorial description of the eigenvalues with multiplicities of the transition matrix is given and the transition

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matrix is shown to be diagonalizable.

The second interesting result concerns the descent algebra of a finite Coxeter group, a subalgebra of the group algebra of the Coxeter group. To any finite Coxeter group is associated a hyperplane arrangement and the Coxeter group acts on the faces of this arrangement. This gives an action of the Coxeter group on the face semigroup algebra of the arrangement. The subalgebra of elements invariant under the action of the Coxeter group is anti-isomorphic to the descent algebra of the Coxeter group [Bidigare, 1997, Brown, 2000]. The descent algebra was introduced in [Solomon, 1976] and the proof that it is indeed an algebra is rather involved. This approach via hyperplane arrangements provides a new, somewhat simpler setting for studying the descent algebra. See [Schocker, 2005].

This chapter presents a study of the algebraic structure of the face semigroup algebra $k\mathcal{F}$ of an arbitrary central hyperplane arrangement in $\mathbb{R}^d$. Throughout k will denote a field of arbitrary characteristic and $\mathcal{A}$ a finite collection of hyperplanes passing through the origin in $\mathbb{R}^d$. The *intersection lattice* of $\mathcal{A}$ is the set $\mathcal{L}$ of intersections of subsets of hyperplanes in $\mathcal{A}$ ordered by inclusion. (Note that some authors order the intersection lattice by reverse inclusion rather than inclusion.)

The chapter begins with some background. Sections 1.2 and 1.3 recall notions from the theory of posets and hyperplane arrangements. Section 1.4 defines the face semigroup algebra of a hyperplane arrangement and describes its irreducible representations.

In Section 1.5 a complete system of primitive orthogonal idempotents $\{e_X\}_{X\in\mathcal{L}}$ in $k\mathcal{F}$ is constructed. This leads to a description of the projective indecomposable $k\mathcal{F}$-modules (Section 1.6) and a computation of the Cartan invariants of $k\mathcal{F}$ (see (1.11)). The projective indecomposable modules are used to construct projective

resolutions of the simple $k\mathcal{F}$-modules in Section 1.7.

Since all the irreducible representations of $k\mathcal{F}$ are 1-dimensional, it follows that $k\mathcal{F}$ is a quotient of the path algebra of a quiver (a directed graph). The quiver with relations of $k\mathcal{F}$ is computed in Section 1.8. The vertices of the quiver correspond to elements of the intersection lattice $\mathcal{L}$. There is exactly one arrow $X \to Y$ if $Y \lessdot X$, and these are the only arrows of the quiver. There is exactly one relation for each interval of length two in $\mathcal{L}$: the sum of the paths of length two in the interval. This implies that $k\mathcal{F}$ depends only on the intersection lattice $\mathcal{L}$ of the hyperplane arrangement.

Section 1.9 proves that $k\mathcal{F}$ is a Koszul algebra and computes the Ext-algebra (or Koszul dual) of $k\mathcal{F}$. It is the incidence algebra $I(\mathcal{L}^*)$ of the opposite poset $\mathcal{L}^*$ of $\mathcal{L}$. This is used in Section 1.9.4 to compute the Hochschild homology and cohomology of $k\mathcal{F}$.

Section 1.10 explores connections with poset cohomology. The face semigroup algebra decomposes into a direct sum of subspaces that are isomorphic to the *order cohomology* $H^*([X,Y])$ of intervals $[X,Y]$ of the intersection lattice $\mathcal{L}$. A new cohomology construction is introduced to explain the inherited algebraic structure on this direct sum of cohomology groups. The resulting cohomology algebra, with its cohomology cup product, is isomorphic to $k\mathcal{F}$. Finally, the Whitney cohomology of the geometric lattice $\mathcal{L}^*$ is shown to embed into $k\mathcal{F}$ as the projective indecomposable $k\mathcal{F}$-module spanned by the chambers of the arrangement.

1.2 Posets

This section presents some of the necessary background from the theory of posets. An excellent reference for posets is Chapter 3 of [Stanley, 1997].

A *poset* is a finite set P together with a partial order $\leq$. The *opposite poset* P^* of a poset P is the set P with partial order defined by $x \leq y$ in P^* iff $x \geq y$ in P. For $x, y \in P$, write $x \lessdot y$ and say y *covers* x or x is *covered by* y if $x < y$ and there does not exist $z \in P$ with $x < z < y$. The *Hasse diagram* of P is the graph with exactly one vertex for each x in P and exactly one edge between x and y iff $x \lessdot y$ or $y \lessdot x$. An edge of the Hasse diagram is called a *cover relation.*

A *chain* in P is a sequence of elements $x_0 < x_1 < \cdots < x_r$ in P. A chain $x_0 < x_1 < \cdots < x_r$ is *unrefinable* if $x_{i-1} \lessdot x_i$ for all $1 \leq i \leq r$. The *length* of the chain $x_0 < x_1 < \cdots < x_r$ is r. The *length* or *rank* of a poset is the length of the longest chain in P. For $x \leq y$ in P the *interval* between x and y is the set $[x, y] = \{z \in P \mid x \leq z \leq y\}$. The interval $[x, y]$ is a poset and its rank is denoted by $\ell([x, y])$.

A (finite) poset L is a *lattice* if every pair of elements x, y in L has a least upper bound (called *join*) $x \vee y$ and a greatest lower bound (called *meet*) $x \wedge y$ (with respect to the relation $\leq$). There exists an element $\hat{0}$ called the *bottom* of L satisfying $\hat{0} \leq x$ for all $x \in L$. Similarly, there exists an element $\hat{1}$ in L called the *top* of L satisfying $x \leq \hat{1}$ for all $x \in L$.

The *Möbius function* μ of a finite poset P is defined recursively by the equations

$$\mu(x, x) = 1 \quad \text{and} \quad \mu(x, y) = - \sum_{x \leq z < y} \mu(x, z),$$

for all $x < y$ in P. If $x \not\leq y$, then set $\mu(x, y) = 0$. The *Möbius inversion formula* [Stanley, 1997, §3.7] states that $g(x) = \sum_{y \leq x} f(y)$ iff $f(x) = \sum_{y \leq x} g(y)\mu(y, x)$, where $f, g : P \to \mathbb{R}$.

1.3 Hyperplane Arrangements

This section recalls some definitions from the theory of hyperplane arrangements (see [Orlik and Terao, 1992]).

1.3.1 Hyperplane Arrangements

A *hyperplane arrangement* $\mathcal{A}$ in $\mathbb{R}^d$ is a finite set of hyperplanes in $\mathbb{R}^d$. We restrict our attention to *central* hyperplane arrangements where all the hyperplanes contain $0 \in \mathbb{R}^d$. Each hyperplane $H \in \mathcal{A}$ determines two open half-spaces of $\mathbb{R}^d$ denoted H^+ and H^-. The choice of which half-space to label $+$ or $-$ is arbitrary, but fixed.

1.3.2 The Face Poset

A *face* of $\mathcal{A}$ is a nonempty intersection of the form

$$x = \bigcap_{H \in \mathcal{A}} H^{\sigma_H},$$

where $\sigma_H \in \{+, -, 0\}$ and $H^0 = H$. The sequence $\sigma(x) = (\sigma_H)_{H \in \mathcal{A}}$ is the *sign sequence* of x. A *chamber* c is a face such that $\sigma_H(c) \neq 0$ for all $H \in \mathcal{A}$.

The *face poset* $\mathcal{F}$ of $\mathcal{A}$ is the set of faces of $\mathcal{A}$ partially ordered by

$$x \leq y \iff \text{for each } H \in \mathcal{A} \text{ either } \sigma_H(x) = 0 \text{ or } \sigma_H(x) = \sigma_H(y).$$

Equivalently, $x \leq y \iff x \subset \bar{y}$. If $x \leq y$, then we say *x is a face of y*. Note that the chambers are the maximal elements in this partial order.

1.3.3   The Support Map and the Intersection Lattice

The *support* of a face $x \in \mathcal{F}$ is the the intersection of the hyperplanes in $\mathcal{A}$ containing x,

$$\mathrm{supp}(x) = \bigcap_{\substack{H \in \mathcal{A} \\ \sigma_H(x)=0}} H.$$

The set $\mathcal{L} = \mathrm{supp}(\mathcal{F})$ of supports of faces of $\mathcal{A}$ is a graded lattice ordered by inclusion, called the *intersection lattice* of $\mathcal{A}$. (Some authors order the intersection lattice by *reverse* inclusion, so some care is needed while reading the literature.) The rank of $X \in \mathcal{L}$ is the dimension of the subspace $X \subset \mathbb{R}^d$ if the intersection of all the hyperplanes in the arrangement is trivial. For $X, Y \in \mathcal{L}$ the *meet* $X \wedge Y$ of X and Y is the intersection $X \cap Y$ and the *join* $X \vee Y$ of X and Y is $X + Y$, the smallest subspace of $\mathbb{R}^d$ containing X and Y. The opposite poset $\mathcal{L}^*$ of $\mathcal{L}$ is a geometric lattice. The top element $\hat{1}$ of $\mathcal{L}$ is the ambient vector space $\mathbb{R}^d$ and the bottom element $\hat{0}$ is the intersection of all hyperplanes in the arrangement $\bigcap_{H \in \mathcal{A}} H$. The chambers are the faces of support $\hat{1}$. Since $\mathrm{supp}(x) \leq \mathrm{supp}(y)$ if $x \leq y$, the support map $\mathrm{supp} : \mathcal{F} \to \mathcal{L}$ is an order-preserving poset surjection.

1.3.4   Deletion and Restriction

Fix $X \in \mathcal{L}$. The faces y of $\mathcal{A}$ with $\mathrm{supp}(y) \leq X$ are the faces of the arrangement $\mathcal{A}_X = \{H \cap X \mid X \not\subseteq H \in \mathcal{A}\}$. $\mathcal{A}_X$ is the *restriction* to X and the face poset of $\mathcal{A}_X$ is denoted by $\mathcal{F}_{\leq X}$. The intersection lattice $\mathcal{L}_{\leq X}$ of $\mathcal{A}_X$ is the interval $[\hat{0}, X]$ of $\mathcal{L}$.

Given $X \in \mathcal{L}$ let $\mathcal{A}^X = \{H \in \mathcal{A} \mid X \subset H\}$ denote the set of hyperplanes in $\mathcal{A}$ containing X. $\mathcal{A}^X$ is a *deletion* of $\mathcal{A}$. If $x \in \mathcal{F}$ with $\mathrm{supp}(x) = X$, then the face poset $\mathcal{F}^X$ of $\mathcal{A}^X$ is isomorphic to the subposet of $\mathcal{F}$ of all faces having x as a face:

$\mathcal{F}^X \cong \{y \in \mathcal{F} \mid x \le y\}$. The intersection lattice of $\mathcal{A}^X$ is the interval $[X, \hat{1}] \subset \mathcal{L}$.

1.4 The Face Semigroup Algebra

This section recalls the semigroup structure on the faces of a hyperplane arrangement and the irreducible representations of the resulting semigroup algebra. See [Brown, 2000] for details.

1.4.1 The Face Semigroup

For $x, y \in \mathcal{F}$ the product xy is the face of $\mathcal{A}$ with sign sequence

$$\sigma_H(xy) = \begin{cases} \sigma_H(x), & \text{if } \sigma_H(x) \ne 0, \\[2mm] \sigma_H(y), & \text{if } \sigma_H(x) = 0. \end{cases}$$

This product is associative and noncommutative with identity element the intersection of all the hyperplanes in the arrangement $1 = \bigcap_{H \in \mathcal{A}} H$. Note that the support of the identity element 1 is $\hat{0}$ (and not $\hat{1}$). The support map $\text{supp} : \mathcal{F} \to \mathcal{L}$ satisfies $\text{supp}(xy) = \text{supp}(x) \vee \text{supp}(y)$ for all $x, y \in \mathcal{F}$. Therefore supp is a semigroup surjection, where $\mathcal{L}$ is considered a semigroup with product given by join $\vee$, as well as an ordering-preserving poset surjection.

Remark 1.1. There is a nice geometric interpretation of this product. The face xy is the face that one enters by moving a *small* positive distance along any straight line from x to y.

Proposition 1.2. *For all $x, y \in \mathcal{F}$,*

1. *$x^2 = x$,*

2. *$xyx = xy$,*

3. $xy = y$ *iff* $x \leq y$,

4. $xy = x$ *iff* $\operatorname{supp}(y) \leq \operatorname{supp}(x)$,

5. $\operatorname{supp}(xy) = \operatorname{supp}(x) \vee \operatorname{supp}(y)$,

Remark 1.3. Conditions (1) and (2) of the proposition say that $\mathcal{F}$ is a *left regular band*.

1.4.2 The Face Semigroup Algebra

The *face semigroup algebra* of $\mathcal{A}$ with coefficients in the field k is the semigroup algebra $k\mathcal{F}$ of the face semigroup $\mathcal{F}$ of $\mathcal{A}$. Explicitly, it consists of linear combinations of elements of $\mathcal{F}$ with multiplication induced by the product of $\mathcal{F}$. The face semigroup algebra $k\mathcal{F}$ is a finite dimensional associative algebra with identity $1 = \bigcap_{H \in \mathcal{A}} H$.

Unless explicitly stated otherwise, no assumptions will be made on the characteristic of the field k.

1.4.3 Irreducible Representations

This section summarizes Section 7.2 of [Brown, 2000] constructing the irreducible representations of $k\mathcal{F}$.

Since $\mathcal{F}$ and $\mathcal{L}$ are semigroups, the support map $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$ extends linearly to a surjection of algebras $\operatorname{supp} : k\mathcal{F} \to k\mathcal{L}$. The kernel of this map is nilpotent and the semigroup algebra $k\mathcal{L}$ is isomorphic to a product of copies of the field k, one copy for each element of $\mathcal{L}$. This implies that $\ker(\operatorname{supp})$ is the Jacobson radical of $k\mathcal{F}$ and that the irreducible representations of $k\mathcal{F}$ are given by the components of the composition $k\mathcal{F} \xrightarrow{\operatorname{supp}} k\mathcal{L} \xrightarrow{\cong} \prod_{X \in \mathcal{L}} k$. This last map sends $X \in \mathcal{L}$ to the

vector with 1 in the Y-position if $Y \geq X$ and 0 otherwise. The X-component of this surjection is the map $\chi_X : k\mathcal{F} \to k$ defined on the faces $y \in \mathcal{F}$ by

$$\chi_X(y) = \begin{cases} 1, & \text{if supp}(y) \leq X, \\ 0, & \text{otherwise.} \end{cases}$$

The elements

$$E_X = \sum_{Y \geq X} \mu(X, Y)Y, \tag{1.4.1}$$

one for each $X \in \mathcal{L}$, correspond to the standard basis vectors of $\prod_{X \in \mathcal{L}} k$ under the isomorphism $k\mathcal{L} \cong \prod_{X \in \mathcal{L}} k$ above. They form a basis of $k\mathcal{L}$ and also form a *complete system of primitive orthogonal idempotents* (see Section 1.5).

1.5 Primitive Idempotents

Let A be a k-algebra. An element $e \in A$ is *idempotent* if $e^2 = e$. It is a *primitive idempotent* if e is idempotent and we cannot write $e = e_1 + e_2$ where e_1 and e_2 are nonzero idempotents in A with $e_1 e_2 = 0 = e_2 e_1$. Equivalently, e is primitive iff Ae is an indecomposable A-module. A set of elements $\{e_i\}_{i \in I} \subset A$ is a *complete system of primitive orthogonal idempotents* for A if e_i is a primitive idempotent for every i, if $e_i e_j = 0$ for $i \neq j$ and if $\sum_i e_i = 1$. If $\{e_i\}_{i \in I}$ is a complete system of primitive orthogonal idempotents for A, then $A \cong \bigoplus_{i \in I} Ae_i$ as left A-modules and $A \cong \bigoplus_{i,j \in I} e_i A e_j$ as k-vector spaces.

1.5.1 A Complete System of Primitive Orthogonal Idempotents

For each $X \in \mathcal{L}$, fix an $x \in \mathcal{F}$ with $\operatorname{supp}(x) = X$ and define elements in $k\mathcal{F}$ recursively by the formula,

$$e_X = x - \sum_{Y > X} x e_Y. \tag{1.5.1}$$

Note that $e_{\hat{1}}$ is an arbitrarily chosen chamber.

Lemma 1.4. *Let $w \in \mathcal{F}$ and $X \in \mathcal{L}$. If $\operatorname{supp}(w) \not\leq X$, then $we_X = 0$.*

Proof. We proceed by induction on X. This is vacuously true if $X = \hat{1}$. Suppose the result holds for all $Y \in \mathcal{L}$ with $Y > X$. Suppose $w \in \mathcal{F}$ and $W = \operatorname{supp}(w) \not\leq X$. Using the definition of e_X and the identity $wxw = wx$ (Proposition 1.2 (2)),

$$we_X = wx - \sum_{Y > X} wx e_Y = wx - \sum_{Y > X} wx(we_Y).$$

By induction, $we_Y = 0$ if $W \not\leq Y$. Therefore, the summation runs over Y with $W \leq Y$. But $Y > X$ and $Y \geq W$ iff $Y \geq W \vee X$, so the summation runs over Y with $Y \geq W \vee X$.

$$we_X = wx - \sum_{Y > X} wx(we_Y) = wx - \sum_{Y \geq X \vee W} wx e_Y.$$

Now let z be the element of support $X \vee W$ chosen in defining $e_{X \vee W}$. So $e_{X \vee W} = z - \sum_{Y > X \vee W} z e_Y$. Note that $z e_{X \vee W} = e_{X \vee W}$ since $z = z^2$. Therefore, $z = \sum_{Y \geq X \vee W} z e_Y$. Since $\operatorname{supp}(wx) = W \vee X = \operatorname{supp}(z)$, it follows from Proposition 1.2 (4) that $wx = wxz$. Combining the last two statements,

$$we_X = wx - \sum_{Y \geq X \vee W} wx e_Y = wx \left(z - \sum_{Y \geq X \vee W} z e_Y \right) = 0. \qquad \square$$

Theorem 1.5. *The elements $\{e_X\}_{X \in \mathcal{L}}$ form a complete system of primitive orthogonal idempotents in $k\mathcal{F}$.*

Proof. Complete. $1 = \bigcap_{H \in \mathcal{A}} H$ is the only element of support $\hat{0}$. Hence, $e_{\hat{0}} = 1 - \sum_{Y > \hat{0}} e_Y$. Therefore,

$$\sum_X e_X = e_{\hat{0}} + \sum_{X \neq \hat{0}} e_X = \left(1 - \sum_{X > \hat{0}} e_X \right) + \sum_{X \neq \hat{0}} e_X = 1.$$

Idempotent. Since e_Y is a linear combination of elements of support at least Y, $e_Y z = e_Y$ for any z with $\text{supp}(z) \leq Y$ (Proposition 1.2 (4)). Using the definition of e_X, the facts $e_X = x e_X$ and $e_Y = e_Y y$, and Lemma 1.4,

$$e_X^2 = \left(x - \sum_{Y > X} x e_Y \right) e_X = x e_X - \sum_{Y > X} x e_Y (y e_X) = x e_X = e_X.$$

Orthogonal. We show that for every $X \in \mathcal{L}$, $e_X e_Y = 0$ for $Y \neq X$. If $X = \hat{1}$, then $e_X e_Y = e_X x e_Y = 0$ for every $Y \neq X$ by Lemma 1.4 since $X = \hat{1}$ implies $X \not\leq Y$. Now suppose the result holds for $Z > X$. That is, $e_Z e_Y = 0$ for all $Y \neq Z$. If $X \not\leq Y$, then $e_X e_Y = 0$ by Lemma 1.4. If $X < Y$, then $e_X e_Y = x e_Y - \sum_{Z > X} x(e_Z e_Y) = x e_Y - x e_Y^2 = 0$.

Primitive. We'll show that e_X lifts $E_X = \sum_{Y \geq X} \mu(X, Y) Y$ (see equation (1.4.1)) for all $X \in \mathcal{L}$, a primitive idempotent in $k\mathcal{L}$. If $X = \hat{1}$, then $\text{supp}(e_{\hat{1}}) = \hat{1} = E_{\hat{1}}$. Suppose the result holds for $Y > X$. Then $\text{supp}(e_X) = \text{supp}(x - \sum_{Y > X} x e_Y) = X - \sum_{Y > X} (X \vee E_Y)$. Since E_Y is a linear combination of elements $Z \geq Y$, it follows that $X \vee E_Y = E_Y$ if $Y > X$. Therefore, $\text{supp}(e_X) = X - \sum_{Y > X} E_Y$. The Möbius inversion formula applied to $E_X = \sum_{Y \geq X} \mu(X, Y) Y$ gives $X = \sum_{Y \geq X} E_X$. Hence, $\text{supp}(e_X) = X - \sum_{Y > X} E_Y = E_Y$.

To see that this is sufficient, suppose E is a primitive idempotent in $k\mathcal{L}$ and that e is an idempotent lifting E. Suppose $e = e_1 + e_2$ with e_i orthogonal idempotents.

Then $E = \operatorname{supp}(e) = \operatorname{supp}(e_1) + \operatorname{supp}(e_2)$. Since E is primitive and $\operatorname{supp}(e_1)$ and $\operatorname{supp}(e_2)$ are orthogonal idempotents, $\operatorname{supp}(e_1) = 0$ or $\operatorname{supp}(e_2) = 0$. Say $\operatorname{supp}(e_1) = 0$. Then e_1 is in the kernel of supp. This kernel is nilpotent so $e_1^n = 0$ for some $n \geq 0$. Hence $e_1 = e_1^n = 0$. Therefore, e is a primitive idempotent. $\qquad\square$

Remark 1.6. We can replace $x \in \mathcal{F}$ in (1.5.1) with any linear combination $\tilde{x} = \sum_{\operatorname{supp}(x)=X} \lambda_x x$ of elements of support X whose coefficients λ_x sum to 1. The proofs still hold since the element $\tilde{x}$ is idempotent and satisfies $\operatorname{supp}(\tilde{x}) = X$ and $\tilde{x}y = \tilde{x}$ if $\operatorname{supp}(y) \leq X$. Unless explicitly stated we will use the idempotents constructed above.

1.5.2 A Basis of Primitive Idempotents

Proposition 1.7. *The set $\{xe_{\operatorname{supp}(x)} \mid x \in \mathcal{F}\}$ is a basis of $k\mathcal{F}$ of primitive idempotents.*

Proof. Let $y \in \mathcal{F}$. Then by Corollary 1.5 and Lemma 1.4,

$$y = y1 = y\sum_X e_X = \sum_{X \geq \operatorname{supp}(y)} ye_X = \sum_{X \geq \operatorname{supp}(y)} (yx)e_X.$$

Since $\operatorname{supp}(yx) = \operatorname{supp}(y) \vee \operatorname{supp}(x) = X$, the face y is a linear combination of the elements of the form $xe_{\operatorname{supp}(x)}$. So these elements span $k\mathcal{F}$. Since the number of these elements is the cardinality of $\mathcal{F}$, which is the dimension of $k\mathcal{F}$, the set forms a basis of $k\mathcal{F}$. The elements are idempotent since $(xe_X)^2 = (xe_X)(xe_X) = xe_X^2 = xe_X$ (since $xyx = xy$ for all $x, y \in \mathcal{F}$). Since xe_X also lifts the primitive idempotent $E_X = \sum_{Y \geq X} \mu(X,Y)Y \in k\mathcal{L}$, it is also a primitive idempotent (see the end of the proof of Corollary 1.5). $\qquad\square$

1.6 Projective Indecomposable Modules

This section constructs the projective indecomposable $k\mathcal{F}$-modules and computes the Cartan invariants of $k\mathcal{F}$.

1.6.1 Projective Indecomposable Modules

For $X \in \mathcal{L}$, let $\mathcal{F}_X \subset \mathcal{F}$ denote the set of faces of support X. For $y \in \mathcal{F}$ and $x \in \mathcal{F}_X$ let

$$y \cdot x = \begin{cases} yx, & \operatorname{supp}(y) \leq \operatorname{supp}(x), \\ 0, & \operatorname{supp}(y) \not\leq \operatorname{supp}(x). \end{cases}$$

Then $k\mathcal{F}_X$ is a $k\mathcal{F}$-module.

Lemma 1.8. *Let $X \in \mathcal{L}$. Then $\{ye_X \mid \operatorname{supp}(y) = X\}$ is a basis for $k\mathcal{F}e_X$.*

Proof. Suppose $\sum_{w \in \mathcal{F}} \lambda_w we_X \in k\mathcal{F}e_X$. If $\operatorname{supp}(w) \not\leq X$, then $we_X = 0$. So suppose $\operatorname{supp}(w) \leq X$. Then $\operatorname{supp}(wx) = \operatorname{supp}(w) \vee X = X$. Therefore,

$$\sum_{w \in \mathcal{F}} \lambda_w we_X = \sum_{w \in \mathcal{F}} \lambda_w (wx)e_X \in \operatorname{span}_k\{ye_X \mid \operatorname{supp}(y) = X\},$$

where x is the element chosen in the construction of e_X (recall that $e_X = xe_X$ since $x^2 = x$). So the elements span $k\mathcal{F}e_X$. These elements are linearly independent being a subset of a basis of $k\mathcal{F}$ (Proposition 1.7). $\qquad\square$

Proposition 1.9. *The $k\mathcal{F}$-modules $k\mathcal{F}_X$ are all the projective indecomposable $k\mathcal{F}$-modules. The radical of $k\mathcal{F}_X$ is $\operatorname{span}_k\{y - y' \mid y, y' \in \mathcal{F}_X\}$.*

Proof. Define a map $\phi : k\mathcal{F}_X \to k\mathcal{F}e_X$ by $w \mapsto we_X$. Then ϕ is surjective since $\phi(y) = ye_X$ for $y \in \mathcal{F}_X$ and since $\{ye_X \mid \operatorname{supp}(y) = X\}$ is basis for $k\mathcal{F}e_X$ (Lemma

1.8). Since $\dim k\mathcal{F}_X = \#\mathcal{F}_X = \dim k\mathcal{F}e_X$, the map ϕ is an isomorphism of k-vector spaces.

ϕ *is a* $k\mathcal{F}$-*module map.* Let $y \in \mathcal{F}$ and let $x \in \mathcal{F}_X$. If $\mathrm{supp}(y) \leq X$, then $\phi(y \cdot x) = \phi(yx) = yxe_X = y\phi(x)$. If $\mathrm{supp}(y) \not\leq X$, then $y \cdot x = 0$. Hence, $\phi(x \cdot y) = 0$. Also, since $\mathrm{supp}(y) \not\leq X$, it follows that $ye_X = 0$. Therefore, $y\phi(x) = yxe_X = yx(ye_X) = yx0 = 0$. So $\phi(y \cdot x) = y\phi(x)$. Hence ϕ is an isomorphism of $k\mathcal{F}$-modules. Since $k\mathcal{F}e_X$ are all the projective indecomposable $k\mathcal{F}$-modules, so are the $k\mathcal{F}_X$. $\qquad\square$

1.6.2 Cartan Invariants

Let $\{e_X\}_{X \in I}$ be a complete system of primitive orthogonal idempotents for a finite dimensional k-algebra A. The *Cartan invariants* of A are defined to be the numbers

$$c_{X,Y} = \dim \mathrm{Hom}_A(Ae_X, Ae_Y).$$

The invariant $c_{X,Y}$ is the multiplicity of the simple module $S_X = (A/\mathrm{rad}A)e_X$ as a composition factor of Ae_Y. The *Cartan matrix* of A is the matrix $[c_{X,Y}]$.

The following is Theorem 1.7.3 of [Benson, 1998].

Theorem 1.10 (Idempotent Refinement Theorem). *Let N by a nilpotent ideal in a ring R and let e be an idempotent in R/N. Then any two idempotents in R lifting e are conjugate in R.*

Proposition 1.11. *For $X, Y \in \mathcal{L}$,*

$$\dim_k \mathrm{Hom}_{k\mathcal{F}}(k\mathcal{F}e_X, k\mathcal{F}e_Y) = |\mu(X,Y)|,$$

where μ is the Möbius function of $\mathcal{L}$. Therefore the Cartan invariants of $k\mathcal{F}$ are $c_{X,Y} = |\mu(X,Y)|$ and the Cartan matrix is triangular of determinant 1.

Proof. Since $\text{Hom}_{k\mathcal{F}}(k\mathcal{F}e_X, k\mathcal{F}e_Y) \cong e_X k\mathcal{F}e_Y$, it follows that $c_{X,Y} = \dim e_X k\mathcal{F}e_Y$. We will use Zaslavsky's Theorem [Zaslavsky, 1975]: The number of chambers in a hyperplane arrangement is $\sum_{X \in \mathcal{L}} |\mu(X, \mathbb{R}^d)|$.

For each $W \in \mathcal{L}$, let w denote an element of support W. If $W \geq X$, then $\text{supp}(xw) = W$, so replace w with xw and construct idempotents e_W as in section 1.5.1. (By the *idempotent refinement theorem* above, it does not matter which lifts of the idempotents in $k\mathcal{L}$ we use to compute the Cartan invariants: $e_X k\mathcal{F}e_Y \cong \tilde{e}_X k\mathcal{F}\tilde{e}_Y$ if e_X and $\tilde{e}_X$ are conjugate and if e_Y and $\tilde{e}_Y$ are conjugate.) Then for each $W \geq X$ we have $xe_W = e_W$, so $x = x\sum_W e_W = x\sum_{W \geq X} e_W = \sum_{W \geq X} e_W$. This gives the equality

$$k(x\mathcal{F}) = xk\mathcal{F} = \sum_{W \geq X} e_W k\mathcal{F}. \tag{1.6.1}$$

Note that $x\mathcal{F}$ is the face poset of the hyperplane arrangement $\mathcal{A}^X = \{H \in \mathcal{A} \mid X \subset H\}$ and that the faces of support Y in $\mathcal{A}^X$ are the chambers in the restricted arrangement $(\mathcal{A}^X)_Y$ (see Section 1.3.4). Zaslavsky's Theorem applied to $(\mathcal{A}^X)_Y$ gives the number of faces of support Y in $\mathcal{A}^X$ is $\sum_{W \in [X,Y]} |\mu(W, Y)|$ since the intersection lattice of $(\mathcal{A}^X)_Y$ is the interval $[X, Y]$ in $\mathcal{L}$. But the number of faces of support Y in $(\mathcal{A}^X)_Y$ is the cardinality of the set $x\mathcal{F}_Y$, which is the dimension of $k(x\mathcal{F}_Y) \cong xk\mathcal{F}_Y \cong xk\mathcal{F}e_Y \cong \bigoplus_{X \leq W \leq Y} e_W k\mathcal{F}e_Y$ by (1.6.1) and Lemma 1.4. Therefore for each $X, Y \in \mathcal{L}$,

$$\sum_{X \leq W \leq Y} \dim e_W k\mathcal{F}e_Y = \sum_{X \leq W \leq Y} |\mu(W, Y)|.$$

The result now follows by induction. If $X = Y$, then $\dim e_X k\mathcal{F}e_X = |\mu(X, X)|$. Suppose the result holds for all W with $X < W \leq Y$. Then

$$\dim e_X k\mathcal{F}e_Y = \sum_{X \leq W \leq Y} |\mu(W, Y)| - \sum_{X < W \leq Y} \dim e_W k\mathcal{F}e_Y$$

$$= \sum_{X \leq W \leq Y} |\mu(W, Y)| - \sum_{X < W \leq Y} |\mu(W, Y)|$$

$$= |\mu(X, Y)|. \qquad \qquad \square$$

1.7 Projective Resolutions of the Simple Modules

1.7.1 A Projective Resolution of the Simple Module Corresponding to $\hat{1}$

In Section 5C of [Brown and Diaconis, 1998] an exact sequence of $k\mathcal{F}$-modules is constructed to compute the multiplicities of the eigenvalues of random walks on the chambers of a hyperplane arrangement. This construction in combination with the above description of the projective indecomposable $k\mathcal{F}$-modules yields a projective resolution of the simple $k\mathcal{F}$-modules.

Let $\mathcal{F}_p \subset \mathcal{F}$ denote the set of faces of codimension p. For $x \in \mathcal{F}$ and $y \in \mathcal{F}_p$, let

$$x \cdot y = \begin{cases} xy, & \mathrm{supp}(x) \leq \mathrm{supp}(y), \\ \\ 0, & \mathrm{supp}(x) \nleq \mathrm{supp}(y). \end{cases}$$

Fix an orientation ϵ_X for every subspace $X \in \mathcal{L}$. If x is a codimension one face of y, then pick a positively oriented basis $\{e_1, \ldots, e_i\}$ of $X = \mathrm{supp}(x)$ and a vector v in y and put

$$[x : y] = \epsilon_Y(e_1, \cdots, e_i, v),$$

where $Y = \mathrm{supp}(y)$. Since X is a codimension one subspace of Y, the mapping $v \mapsto \epsilon_Y(e_1, \cdots, e_i, v)$ is constant on the open halfspaces of Y determined by X. This implies the identity,

$$[x : y] = [\tilde{x} : \tilde{x}y], \text{ if } \mathrm{supp}(\tilde{x}) = \mathrm{supp}(x). \qquad (1.7.1)$$

Lemma 1.12 ([Brown and Diaconis, 1998], §5 Lemma 2). *Let $x, y \in \mathcal{F}$ with x of codimension two in y. Then there are exactly two faces w and z in the open interval (x, y) and we have*

$$[x : w][w : y] = -[x : z][z : y].$$

Proposition 1.13 ([Brown and Diaconis, 1998], §5 Lemma 4). *The following is an exact sequence of $k\mathcal{F}$-modules.*

$$\cdots \longrightarrow k\mathcal{F}_p \xrightarrow{\partial_p} \cdots \longrightarrow k\mathcal{F}_1 \xrightarrow{\partial_1} k\mathcal{F}_0 \xrightarrow{\partial_0} k \longrightarrow 0,$$

where the action of $k\mathcal{F}$ on k is given by $w \cdot \lambda = \lambda$ for all $w \in \mathcal{F}$ and $\lambda \in k$. The differential ∂_i is given by $\partial_0(c) = 1$ for all $c \in \mathcal{F}_0$ and for $x \in \mathcal{F}_p$,

$$\partial_p(x) = \sum_{y > x} [x : y] y.$$

Sketch of the proof. It is easy to check that the complex consists of $k\mathcal{F}$-modules and that ∂_i is a $k\mathcal{F}$-module map. It remains to explain why the complex is exact. Suppose that the intersection of all the hyperplanes in a point, otherwise quotient out by that subspace. Intersecting the hyperplane arrangement with a sphere centered at the origin induces a regular cell decomposition Σ of the $(d-1)$-sphere whose cells correspond to the faces $x \neq 1$ of $\mathcal{A}$. The dual of Σ is the boundary of a polytope (a zonotope, actually) Z. Therefore, the poset of nonempty faces of Z is anti-isomorphic to the face poset $\mathcal{F}$ of $\mathcal{A}$. Since Z is contractible any augmented cellular chain complex will be an exact sequence of k-vector spaces. The above complex is precisely the augmented cellular chain complex with incidence numbers given by $[x : y]$. (See [Cooke and Finney, 1967].) Therefore, it is exact. $\square$

Note that $k\mathcal{F}_p \cong \bigoplus_{\text{codim}(X)=p} k\mathcal{F}_X$ as $k\mathcal{F}$-modules and that $k\mathcal{F}_X$ is projective by Proposition 1.9, where $\text{codim}(X)$ is the codimension of the subspace X. So the

$k\mathcal{F}$-modules $k\mathcal{F}_p$ are projective. Also note that in order for ∂_0 to be a $k\mathcal{F}$-module morphism, the action of $k\mathcal{F}$ on k must be given by $\chi_{\hat{1}}$. That is, k is the simple module afforded by the irreducible representation $\chi_{\hat{1}}$. This proves the following result.

Corollary 1.14. *The exact sequence*

$$\cdots \longrightarrow k\mathcal{F}_p \xrightarrow{\partial_p} \cdots \longrightarrow k\mathcal{F}_1 \xrightarrow{\partial_1} k\mathcal{F}_0 \xrightarrow{\partial_0} k \longrightarrow 0$$

is a projective resolution of the simple $k\mathcal{F}$-module afforded by the irreducible representation $\chi_{\hat{1}} : k\mathcal{F} \to k$.

1.7.2 Projective Resolutions of the Simple Modules

Recall that the simple $k\mathcal{F}$-modules are indexed by $X \in \mathcal{L}$, afforded by the representations $\chi_X : k\mathcal{F} \to k$,

$$\chi_X(y) = \begin{cases} 1, & \text{if } \operatorname{supp}(y) \leq X, \\ 0, & \text{otherwise.} \end{cases}$$

Also recall that $\mathcal{F}_{\leq X}$ denotes the face semigroup of $\mathcal{A}_X$, consisting of the set of faces in $\mathcal{F}$ of support contained in X (Section 1.3.4). Let $(\mathcal{F}_{\leq X})_p$ denote the set of faces in $\mathcal{A}_X$ of codimension p in X. Applying the previous result to the hyperplane arrangement $\mathcal{A}_X$ gives a projective resolution

$$\cdots \longrightarrow k(\mathcal{F}_{\leq X})_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k(\mathcal{F}_{\leq X})_1 \xrightarrow{\partial} k\mathcal{F}_X \longrightarrow k_X \longrightarrow 0$$

of the simple $k\mathcal{F}_{\leq X}$-module k_X with action given by $w \cdot \lambda = \lambda$ for all $w \in \mathcal{F}_{\leq X}$ and $\lambda \in k$. The algebra surjection $k\mathcal{F} \to k\mathcal{F}_{\leq X}$ given by $w \mapsto \chi_X(w)w$ for $w \in \mathcal{F}$ puts a $k\mathcal{F}$-module structure on each $k(\mathcal{F}_{\leq X})_p$ and on k. The $k\mathcal{F}$-module structure on k

is precisely that given by $\chi_X : k\mathcal{F} \to k$. Each $k(\mathcal{F}_{\leq X})_p$ is a projective $k\mathcal{F}$-module since the $k\mathcal{F}$-module structure on $k(\mathcal{F}_{\leq X})_p$ decomposes as

$$k(\mathcal{F}_{\leq X})_p \cong \bigoplus_{\substack{Y \leq X, \\ \mathrm{codim}_X(Y) = p}} k\mathcal{F}_Y,$$

where $\mathrm{codim}_X(Y)$ denotes the codimension of Y in X. This establishes the following.

Proposition 1.15. *Let $X \in \mathcal{L}$. Then*

$$\cdots \longrightarrow \left(\bigoplus_{\substack{Y \in \mathcal{L} \\ \mathrm{codim}_X(Y) = p}} k\mathcal{F}_Y \right) \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_X \longrightarrow k_X \longrightarrow 0$$

is a projective resolution of the simple $k\mathcal{F}$-module k_X afforded by $\chi_X : k\mathcal{F} \to k$, where $\partial(w) = \sum_{y > w} [w : y] \chi_X(y) y$ and $\mathrm{codim}_X(Y)$ denotes the codimension of Y in X.

1.8 The Quiver of the Face Semigroup Algebra

1.8.1 The Quiver of a Split Basic Algebra

A finite dimensional k-algebra A is a *(split) basic algebra* if every simple module of A has dimension one. The Ext-*quiver* or just *quiver* Q of a split basic algebra A is a directed graph with one vertex for each isomorphism class of simple modules of A. The number of arrows $x \to y$ is $\dim \mathrm{Ext}_A^1(S_x, S_y)$, where S_x and S_y are simple modules corresponding to the vertices x and y.

A *path* p in Q is a sequence of arrows $x_0 \to x_1 \to \cdots \to x_r$. The path *starts* at $s(p) = x_0$ and *terminates* at $t(p) = x_r$. The length of p is r. Two paths p and q are *parallel* if they start and terminate at the same vertices: $s(p) = s(q)$ and $t(p) = t(q)$. The *path algebra* kQ of a quiver Q is the k-vector space spanned

by the paths in Q with the product of two paths defined by *path composition*: if $p = x_0 \to x_1 \to \cdots \to x_r$ and $q = y_0 \to y_1 \to \cdots \to y_s$, then

$$p \cdot q = \begin{cases} y_0 \to \cdots \to y_s \to x_1 \to \cdots \to x_r, & \text{if } x_0 = s(p) = t(q) = y_s, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Let $P \subset kQ$ be the ideal of kQ generated by the arrows of Q. An ideal $I \subset kQ$ is *admissible* if $I \subset P^2$.

Proposition 1.16 ([Auslander et al., 1995], §III.1 Thereom 1.9). *Let A be a finite dimensional split basic k-algebra with quiver Q. Then $A \cong kQ/I$ where I is an admissible ideal of kQ.*

Let I be an admissible ideal of kQ. An element of I is a *relation* from x to y if it is a k-linear combination of paths in Q beginning at a vertex x and ending at a vertex y. Note that any element $\rho \in I$ can be written as a linear combination of relations since $x\rho y$ is a relation for any pair of vertices $x, y \in Q$. The following result combines Corollary 1.1 and Proposition 1.2 of [Bongartz, 1983]. For the convenience of the reader a proof of this result is included in an appendix to this chapter.

Proposition 1.17. *Let Q be a quiver with no oriented cycles and let I be an admissible ideal. Suppose that R is a minimal set of relations generating I as a two-sided ideal of kQ. Then the number of relations from x to y in R is the dimension of the k-vector space $\mathrm{Ext}^2_{kQ/I}(S_x, S_y)$.*

1.8.2 The Quiver of the Face Semigroup Algebra

Since every simple $k\mathcal{F}$-module is of dimension one, $k\mathcal{F}$ is a split basic algebra. In this section we'll compute the quiver $\mathcal{Q}$ of $k\mathcal{F}$ and in the next section we'll describe

an ideal I such that $kQ/I \cong k\mathcal{F}$.

Lemma 1.18. *For $X, Y \in \mathcal{L}$ and $p \geq 0$,*

$$\operatorname{Ext}^p_{k\mathcal{F}}(k_X, k_Y) \cong \begin{cases} k, & \text{if } Y \leq X \text{ and } \dim(X) - \dim(Y) = p, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\operatorname{codim}_X(W)$ denote the codimension of W in X and let C_p denote $\bigoplus_{\operatorname{codim}_X(W)=p} k\mathcal{F}_W$. Applying the functor $\operatorname{Hom}(-, k_Y)$ to the projective resolution of k_X in Proposition 1.15, gives the cocomplex

$$\cdots \xrightarrow{\partial_p^*} \operatorname{Hom}_{k\mathcal{F}}(C_p, k_Y) \xrightarrow{\partial_{p+1}^*} \operatorname{Hom}_{k\mathcal{F}}(C_{p+1}, k_Y) \xrightarrow{\partial_{p+2}^*} \cdots .$$

Now $\operatorname{Hom}_{k\mathcal{F}}(C_p, k_Y) \cong \bigoplus_{\operatorname{codim}_X(W)=p} \operatorname{Hom}_{k\mathcal{F}}(k\mathcal{F}_W, k_Y)$ and

$$\operatorname{Hom}_{k\mathcal{F}}(k\mathcal{F}_W, k_Y) \cong \operatorname{Hom}_{k\mathcal{F}}(k\mathcal{F}e_W, k_Y) \cong e_W \cdot k_Y = \begin{cases} k, & \text{if } W = Y, \\ 0, & \text{otherwise,} \end{cases}$$

where we used the fact that $\chi_Y(e_W) = 0$ if $W \neq Y$ and 1 otherwise. (If $W \neq Y$, then $\chi_Y(e_Y) = 1$ implies $\chi_Y(e_W) = \chi_Y(e_W)\chi_Y(e_Y) = \chi_Y(e_W e_Y) = 0$.) Since $\operatorname{Hom}_{k\mathcal{F}}(k\mathcal{F}_W, k_Y)$ vanishes unless $W = Y$, the entries in the above cocomplex vanish in all degrees except for that in which $k\mathcal{F}_Y$ appears. This degree is precisely $\operatorname{codim}_X(Y) = \dim(X) - \dim(Y)$, in which case $\operatorname{Hom}_{k\mathcal{F}}(k\mathcal{F}_Y, k_Y) \cong k$. $\square$

Corollary 1.19. *The quiver $\mathcal{Q}$ of $k\mathcal{F}$ is given by the Hasse diagram of the intersection lattice $\mathcal{L}$. The cover relations are oriented by $X \to Y \iff X \gtrdot Y$.*

Proof. The vertices of $\mathcal{Q}$ are in one-to-one correspondence with the isomorphism classes of simple $k\mathcal{F}$-modules. These are indexed by the elements of $\mathcal{L}$. The number of arrows $X \to Y$ is

$$\dim \operatorname{Ext}^1_{k\mathcal{F}}(k_X, k_Y) = \begin{cases} 1, & \text{if } X \gtrdot Y, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

1.8.3 Quiver Relations

This section defines a k-algebra surjection $\varphi : k\mathcal{Q} \to k\mathcal{F}$ and identifies a minimal generating set of the kernel. The kernel is an admissible ideal of the path algebra $k\mathcal{Q}$, so this generating set gives the quiver relations.

1.8.3A. First Version

Let $\partial : k\mathcal{F} \to k\mathcal{F}$ be the map

$$\partial(y) = \sum_{\substack{x \in k\mathcal{F} \\ x > y}} [y : x]x,$$

where $[y : x]$ is the incidence number defined in equation (1.7.1). Define a k-algebra morphism $\varphi : k\mathcal{Q} \longrightarrow k\mathcal{F}$ by

$$\varphi(X) = e_X \text{ for } X \in \mathcal{Q}_0,$$

$$\varphi(X \to Y) = e_Y \partial(y) e_X,$$

$$\varphi(X_0 \to X_1 \to \cdots \to X_r) = \varphi(X_{r-1} \to X_r) \cdots \varphi(X_0 \to X_1),$$

where y was chosen in the construction of e_Y. (Actually, y can be any element of support Y. This follows from the identity $xx' = x$ iff $\mathrm{supp}(x) \geq \mathrm{supp}(x')$.) Using Lemma 1.4 and that $e_Y = y - \sum_{Z>Y} y e_Z$, it follows that $e_Y \partial(y) e_X = ([y : x_1]x_1 + [y : x_2]x_2)e_X$ where x_1 and x_2 are the two faces of support X with common codimension one face y. In particular, this is nonzero.

Proposition 1.20. *Let $\varphi : k\mathcal{Q} \to k\mathcal{F}$ be the map defined above. For each interval $[Z, X]$ of length two in $\mathcal{L}$, the sum of all paths of length two from X to Z*

$$\sum_{Y \in (Z, X)} (X \to Y \to Z)$$

is an element of the kernel of φ. These elements form a minimal generating set of relations for the kernel of φ.

Proof. If R is a minimal set of relations generating $\ker \varphi$, then Proposition 1.17 gives that the number of elements of $Z.R.X$ (the number of relations in R starting at X and ending at Z) is $\dim \text{Ext}^2_{k\mathcal{F}}(k_X, k_Z)$. This is 1 if $[Z, X]$ is an interval of length two and 0 otherwise. Therefore, we need only one relation for each interval of length two in $\mathcal{L}$.

Let z be the element of support Z chosen in the construction of e_Z. Then $\sum_{Y \in (Z,X)} \varphi(X \to Y \to Z)$ is a linear combination of elements of the form $\tilde{x}e_X$ with $\tilde{x}$ of support X having z as a face. If $\tilde{x}$ has z as a face, then z is of codimension two in $\tilde{x}$. Lemma 1.12 gives that $\tilde{x}$ has exactly two codimension one faces $\tilde{y}$ and $\tilde{w}$. Since

$$\varphi(\text{supp}(\tilde{y}) \to Z)\varphi(X \to \text{supp}(\tilde{y}))$$

$$= ([z : \tilde{y}]\tilde{y} + [z : y']y')([y : x_1]x_1 + [y : x_2]x_2)e_X$$

and one of $\tilde{y}x_1$ or $\tilde{y}x_2$ must be $\tilde{x}$ — suppose $\tilde{y}x_1 = \tilde{x}$ — we see that $\tilde{x}e_X$ appears in $\varphi(X \to \text{supp}(\tilde{y}) \to Z)$ with coefficient $[z : \tilde{y}][y : x_1]$. The identity (1.7.1) gives this coefficient is $[z : \tilde{y}][\tilde{y} : \tilde{x}]$. Similarly, $\tilde{x}e_X$ appears in $\varphi(X \to \text{supp}(\tilde{w}) \to Z)$ with coefficient $[z : \tilde{w}][\tilde{w} : \tilde{x}]$. Lemma 1.12 shows that these two coefficients sum to zero. Therefore, $\sum_{Y \in (Z,X)} \varphi(X \to Y \to Z) = 0$. $\square$

Corollary 1.21. *The face semigroup algebra $k\mathcal{F}$ of a hyperplane arrangement depends only on the intersection lattice $\mathcal{L}$.*

Note that this implies that arrangements with the same intersection lattice but nonisomorphic face posets have isomorphic face semigroup algebras.

1.8.3B. Second Version

In this section we note that the idempotents e_X used in the previous section to define φ can be changed slightly without affecting the kernel of φ. This will be used in the next chapter.

For each $X \in \mathcal{L}$ let L_X denote a nonempty set of elements of support X and let $\lambda_X = |L_X|$. In what follows we will need that the characteristic of k does not divide λ_X for all $X \in \mathcal{L}$. Let $\tilde{X}$ denote the sum of the elements in L_X divided by λ_X. Then $\tilde{X}$ is an idempotent and the elements $e_X = \tilde{X} - \sum_{Y>X} \tilde{X} e_Y$ form a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ (see Remark 1.6). Define $\varphi : k\mathcal{Q} \to k\mathcal{F}$ using these idempotents: the image of vertex X is the idempotent e_X; the image of an arrow $X \to Y$ is $e_Y \partial(y) e_X$, where y is any element of support Y.

To see that the kernel of φ is described by Proposition 1.20, let $(X \to Y \to Z)$ be a path in $\mathcal{Q}$ and note that $\varphi(X \to Y \to Z)$ can be written as

$$\frac{1}{\lambda_Z} \sum_{z \in L_Z} \left([z : y_1^z] y_1^z + [z : y_2^z] y_2^z \right) \left([y : x_1^y] x_1^y + [y : x_2^y] x_2^y \right) e_X,$$

where y_1^z and y_2^z are the two faces of support Y with z as a face and x_1^y and x_2^y are the two faces of support X with y as a face. (Use Lemma 1.4; that $e_X = \tilde{X} - \sum_{Y>X} \tilde{X} e_Y$ for all $X \in \mathcal{L}$; and Proposition 1.2.)

Next we will show that the coefficient of $y_i^z x_j^y$ in the above is $\frac{1}{\lambda_Z} [z : y_i^z][y : x_j^y]$. This amounts to showing that if $y_i^z x_j^y = y_{i'}^{z'} x_{j'}^y$, then $z = z'$, $i = i'$ and $j = j'$. Well, both z and z' are faces of $y_i^z x_j^y = y_{i'}^{z'} x_{j'}^y$, but no face can have two distinct faces of the same support. So $z = z'$. Also, y_i^z and $y_{i'}^z$ are faces of $y_i^z x_j^y = y_{i'}^z x_{j'}^y$ of the same support, so in fact $i = i'$. Since $y_i^z x_j^y = y_i^z x_{j'}^y$, it follows that x_j^y and $x_{j'}^y$ are on the same side of Y. But, by definition, they are on different sides of Y. So $j = j'$.

Let $x \in \mathcal{F}$ have support X and suppose $x e_X$ is a summand of $\varphi(X \to Y \to Z)$.

Then $x = y_i^z x_j^y$ for some $i, j \in \{1, 2\}$, $z \in L_Z$. Since there are exactly two faces w_1 and w_2 in the open interval $\{w \in \mathcal{F} : z < w < x\}$, it follows that y_i^z is either w_1 or w_2. In the former case the coefficient of xe_X is

$$\frac{1}{\lambda_Z}[z : w_1][y : x_j^y] = \frac{1}{\lambda_Z}[z : w_1][w_1 y' : w_1 x_j^y] = \frac{1}{\lambda_Z}[z : w_1][w_1 : x],$$

using Equation (1.7.1). Similarly, if $y = w_2$, then the coefficient is $\frac{1}{\lambda_Z}[z : w_2][w_2 : x]$. Therefore, the coefficient of xe_X in $\sum_{X \lessdot Y \lessdot Z} \varphi(X \to Y \to Z)$ is, by Lemma 1.12,

$$\frac{1}{\lambda_Z}[z : w_1][w_1 : x] + \frac{1}{\lambda_Z}[z : w_2][w_2 : x] = 0.$$

So $\sum_{X \lessdot Y \lessdot Z} \varphi(X \to Y \to Z) = 0$ since $\{xe_X : \mathrm{supp}(x) = X\}$ is a basis of $k\mathcal{F}e_X$.

1.9 The Ext-algebra of the Face Semigroup Algebra

1.9.1 Koszul Algebras

Our treatment of Koszul algebras closely follows [Beilinson et al., 1996]. Let k be a field. A k-algebra A is a *graded k-algebra* if there exists a k-vector space decomposition $A \cong \bigoplus_{i \geq 0} A_i$ satisfying $A_i A_j \subset A_{i+j}$. Here $A_i A_j$ is the set of elements $\{\sum_l a_l a_l' \mid a_l \in A_i, a_l' \in A_j\}$. The subspace A_0 is considered an A-module by identifying it with the A-module $A/\bigoplus_{i>0} A_i$.

If $A = \bigoplus_{i \geq 0} A_i$ is a graded k-algebra, then a *graded A-module M* is an A-module with a vector space decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ satisfying $A_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. A graded A-module M is *generated in degree i* if $M_j = 0$ for $j < i$ and $M_j = A_{j-i} M_i$ for all $j \geq i$. If M and N are graded A-modules, then an A-module morphism $f : M \to N$ has *degree p* if $f(M_i) \subset N_{i+p}$ for all i.

A graded A-module M has a *linear resolution* if M admits a projective resolu-

tion

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

with P_i a graded A-module generated in degree i and d_i a degree 0 morphism form all $i \geq 0$. Observe that if M admits a linear resolution, then M is generated in degree 0.

Definition 1.22. A graded k-algebra $A = \bigoplus_{i \geq 0} A_i$ is a *Koszul algebra* if A_0 is a semisimple k-algebra and A_0, considered as a graded A-module concentrated in degree 0, admits a linear resolution.

A *quadratic k-algebra* is a graded k-algebra $A = \bigoplus_{i \geq 0} A_i$ such that A_0 is semisimple and A is generated by A_1 over A_0 with relations of degree 2. Explicitly, $A = \bigoplus_{i \geq 0} A_i$ is quadratic if A_0 is semisimple and A is a quotient of the free tensor algebra $T_{A_0} A_1 = \bigoplus_{i \geq 0} (A_1)^{\otimes i}$ of the A_0-bimodule A_1 by an ideal generated by elements of degree 2: $A \cong T_{A_0} A_1 / \langle R \rangle$ with $R \subset A_1 \otimes_{A_0} A_1$. Here $(A_1)^{\otimes i}$ denotes the i-fold tensor product of A_1 over A_0.

Proposition 1.23 ([Beilinson et al., 1996], Corollary 2.3.3). *Koszul algebras are quadratic.*

Not all quadratic algebras are Koszul algebras. Furthermore, it is not known for which algebras the notions of quadratic and Koszul coincide.

Let $A = T_{A_0} A_1 / \langle R \rangle$ be a quadratic algebra. If V is an A_0-bimodule, let $V^* = \mathrm{Hom}_{A_0}(V, A_0)$. For any subset $W \subset V$, let $W^{\perp} = \{ f \in V^* \mid f(W) = 0 \}$. The algebra

$$A^! = T_{A_0} A_1^* / \langle R^{\perp} \rangle$$

is the *quadratic dual* of A or the *Koszul dual* of A in the case when A is a Koszul algebra. (There is an important technicality. In defining the quadratic dual the

identification $(V_1^* \otimes \cdots \otimes V_n^*) \cong (V_n \otimes \cdots \otimes V_1)^*$ has been made, where $(f_1 \otimes \cdots \otimes f_n)(v_n \otimes \cdots \otimes v_1) = f_n(v_n f_{n-1}(v_{n-1} \cdots f_1(v_1) \cdots))$ for all $f_i \in V_i^*$ and $v_i \in V_i$.)

If A is a graded k-algebra, then the Ext-*algebra* of A is the graded k-algebra $\text{Ext}(A) = \bigoplus_n \text{Ext}^n(A_0, A_0)$ with multiplication given by Yoneda composition.

Theorem 1.24 ([Beilinson et al., 1996], Theorem 2.10.1 and Theorem 2.10.2). *Suppose A is a Koszul algebra. Then the Koszul dual $A^!$ is a Koszul algebra isomorphic to the* opposite *of the* Ext-*algebra* $\text{Ext}(A)$ *of A and $\text{Ext}(\text{Ext}(A)) \cong A$.*

Before proceeding, we record how the quadratic dual of a quadratic algebra arising as the quotient of the path algebra of a quiver is constructed from the quiver and relations. Note that the path algebra kQ of a quiver Q is the free tensor algebra of the k-vector space kQ_1 spanned by the arrows of Q viewed as a bimodule over the k-vector space kQ_0 spanned by the vertices of Q. It follows that $A = kQ/\langle R \rangle \cong T_{kQ_0} kQ_1/\langle R \rangle$ where R is a set of relations of paths of length two. Then the quadratic dual algebra $A^! \cong T_{kQ_0}(kQ_1)^*/\langle R^\perp \rangle \cong kQ^{opp}/\langle R^\perp \rangle$ is a quotient of the path algebra kQ^{opp} of the opposite quiver Q^{opp} of Q and $R^\perp = \{s \in kQ_2^{opp} \mid s^*(r) = 0 \text{ for all } r \in R\}$. Here $(pq)^* : kQ_2 \to k$ for a path pq of length two in Q^{opp} is the function that takes the value 1 on $qp \in Q$ and 0 otherwise. That is, the quiver of $A^!$ is Q^{opp} and the relations are the relations *orthogonal* to R. (This can be derived from the definitions. See also [Green and Martínez-Villa, 1998]).

1.9.2 The Face Semigroup Algebra is a Koszul Algebra

This section establishes that the face semigroup algebra of a hyperplane arrangement admits a grading making it a Koszul algebra. This is done by constructing a linear resolution for the degree 0 component with respect to the grading inherited from the path length grading on the path algebra of the quiver.

Proposition 1.25. *$k\mathcal{F}$ admits a grading making it a Koszul algebra.*

Proof. The k-vector spaces

$$(k\mathcal{F})_i = \bigoplus_{\mathrm{codim}_Y(X)=i} e_X k\mathcal{F} e_Y.$$

define a grading on $k\mathcal{F}$. (This is the grading inherited from the path length grading on the path algebra $k\mathcal{Q}$ of the quiver $\mathcal{Q}$ of $k\mathcal{F}$.) So $k\mathcal{F}$ is a graded k-algebra. The degree 0 component is

$$(k\mathcal{F})_0 = \bigoplus_{\mathrm{codim}_Y(X)=0} e_X k\mathcal{F} e_Y = \bigoplus_{X\in\mathcal{L}} e_X k\mathcal{F} e_X \cong k^{|\mathcal{L}|},$$

hence is semisimple. It remains to show that $k^{|\mathcal{L}|}$ has a linear resolution. It suffices to show that each simple $k\mathcal{F}$-module k_X has a linear resolution since $k^{|\mathcal{L}|} \cong \bigoplus_{X\in\mathcal{L}} k_X$.

Fix $X \in \mathcal{L}$ and consider the projective resolution of the simple $k\mathcal{F}$-module k_X given by Proposition 1.15,

$$\cdots \longrightarrow \left(\bigoplus_{\mathrm{codim}_X(Y)=p} k\mathcal{F} e_Y \right) \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F} e_X \longrightarrow k_X \longrightarrow 0.$$

For each $k\mathcal{F} e_Y$ define k-subspaces

$$(k\mathcal{F} e_Y)_i = \bigoplus_{\mathrm{codim}(W)=i} e_W k\mathcal{F} e_Y.$$

By Lemma 1.4, if $i < \mathrm{codim}(Y)$, then the degree i component of $k\mathcal{F} e_Y$ is 0. For $i = \mathrm{codim}(Y)$, $(k\mathcal{F})_i = e_Y k\mathcal{F} e_Y = \mathrm{span}_k e_Y$ (Lemma 1.4 again). Since e_Y generates $k\mathcal{F} e_Y$ as a $k\mathcal{F}$-module, $k\mathcal{F} e_Y$ is generated in degree $\mathrm{codim}(Y)$. The boundary operator ∂ is a degree 0 morphism: if $e_W w \in e_W k\mathcal{F} e_Y$, then $\deg(e_W w) = \mathrm{codim}(W)$ and the degree of its image $\partial(e_W w) = e_W \partial(w) \in e_W \partial(k\mathcal{F} e_Y) \subset \bigoplus_{\mathrm{codim}_X(Y')=p} e_W k\mathcal{F} e_{Y'}$ is $\mathrm{codim}(W)$. $\qquad\square$

Remark 1.26. Notice that in creating the surjection $\varphi : k\mathcal{Q} \to k\mathcal{F}$ many choices were taken (in constructing the complete system of primitive orthogonal idempotents and in putting orientations on the subspaces in $\mathcal{L}$). These choices affect the grading inherited by $k\mathcal{F}$ from $k\mathcal{Q}$, but the corresponding graded algebras are isomorphic: two gradings on a k-algebra that both give rise to a Koszul algebra give isomorphic graded k-algebras. See Corollary 2.5.2 of [Beilinson et al., 1996].

1.9.3 The Ext-algebra of the Face Semigroup Algebra

In this section we show that the Ext-algebra of $k\mathcal{F}$ is the incidence algebra of the opposite lattice $\mathcal{L}^*$ of the intersection lattice $\mathcal{L}$.

The *incidence algebra* $I(P)$ of a finite poset P is the set of functions on the subset of $P \times P$ of comparable elements $\{(y, x) \in P \times P \mid y \leq x\}$ with multiplication $(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$. The identity element is the Krönecker δ-function. The incidence algebra $I(P)$ is a split basic algebra and the quiver Q of $I(P)$ has P as its set of vertices and exactly one arrow $x \to y$ if $y \lessdot x$. If I denotes the ideal of kQ generated by differences of parallel paths, then $I(P) \cong kQ/I$. This isomorphism is given by mapping a vertex x of Q to the function $y \mapsto \delta(x, y)$, and an arrow $x \to y$ of Q to the function $(u, v) \mapsto \delta(x, u)\delta(y, v)$.

Proposition 1.27. *The Ext-algebra of $k\mathcal{F}$ is the incidence algebra $I(\mathcal{L}^*)$ of the opposite lattice of the intersection lattice $\mathcal{L}$. Equivalently, it is the opposite algebra $I(\mathcal{L})^{opp}$ of the incidence algebra $I(\mathcal{L})$ of $\mathcal{L}$.*

Proof. Since $k\mathcal{F}$ is a Koszul algebra (Proposition 1.25), its Ext-algebra is its Koszul dual algebra (Theorem 1.24), so we compute the Koszul dual of $k\mathcal{F}$.

Let $\mathcal{Q}$ denote the quiver of $k\mathcal{F}$. From Proposition 1.20, $k\mathcal{F} \cong k\mathcal{Q}/\langle R \rangle$ is the quotient of the path algebra $k\mathcal{Q}$ by the ideal generated by the sums of all parallel

paths of length two,

$$R = \left\{ \sum_{Z \in (Y,X)} (X \to Z \to Y) : X, Y \in \mathcal{L} \right\}.$$

Then $(k\mathcal{F})^! \cong k\mathcal{Q}^{opp}/\langle R^\perp \rangle$ where $R^\perp$ is spanned by differences of parallel paths of length two in $\mathcal{Q}^{opp}$,

$$R^\perp = \{(X \to Z \to Y) - (X \to Z' \to Y) : X \lessdot Z, Z' \lessdot Y \in \mathcal{L}\}.$$

(See the discussion at the end of Section 1.9.1.)

Let $I(\mathcal{L}^*)$ denote the incidence algebra of $\mathcal{L}^*$. Then $I(\mathcal{L}^*) \cong k\mathcal{Q}^{opp}/I$, where I is the ideal generated by differences of parallel paths (not necessarily of length two). Therefore, the proof is complete once it is shown that $R^\perp$ generates I.

If $p : X \to X_1 \to \cdots \to X_n \to Y$ and $q : X \to Y_1 \to \cdots \to Y_n \to Y$ are parallel paths in $\mathcal{Q}$ such that there exists an i with $X_j = Y_j$ for all $j \neq i$, then $p - q \in I$. If there exists a sequence of paths $p = p_0, p_1, \ldots, p_j = q$ with p_{i-1} and p_i differing in exactly one place for $1 \leq i \leq j$, then $p - q = (p_0 - p_1) + \cdots + (p_{j-1} - p_j) \in I$. Therefore, $I = \langle R^\perp \rangle$ if any path in $\mathcal{Q}^{opp}$ can be obtained from any other path that is parallel to it by swapping one vertex at a time (without breaking the path). This follows from the semimodularity of $\mathcal{L}^*$ and by induction on the length of paths in $\mathcal{Q}^{opp}$. Recall that a finite lattice L is *(upper) semimodular* if for every x and y in L, if x and y cover $x \wedge y$, then $x \vee y$ covers x and y.

Let $X \to X_1 \to \cdots \to X_n \to Y$ and $X \to Y_1 \to \cdots \to Y_n \to Y$ be parallel paths in $\mathcal{Q}^{opp}$. Since X_n and Y_n cover $X_n \wedge Y_n = Y$, semimodularity of $\mathcal{L}^*$ gives that $X_n \vee Y_n$ covers both X_n and Y_n. Since $X \leq X_n$ and $X \leq Y_n$, it follows that $X \leq (X_n \vee Y_n)$. So there exists a path from X to $X_n \vee Y_n$. We are now in the

following situation.
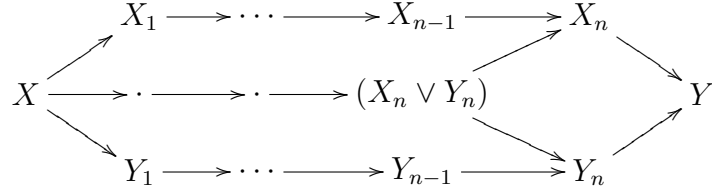
Induction on the length of paths gives that

$$(Y \to \cdots \to Y_{n-1} \to Y_n \to Y) - (X \to \cdots \to (X_n \vee Y_n) \to Y_n \to Y),$$

$$(X \to \cdots \to X_{n-1} \to X_n \to Y) - (X \to \cdots \to (X_n \vee Y_n) \to X_n \to Y)$$

are in $\langle R^{\perp} \rangle$. Clearly,

$$(X \to \cdots \to (X_n \vee Y_n) \to X_n \to Y)$$
$$- (X \to \cdots \to (X_n \vee Y_n) \to Y_n \to Y) \in \langle R^{\perp} \rangle.$$

Therefore,

$$(Y \to \cdots \to Y_{n-1} \to Y_n \to Y) - (X \to \cdots \to X_{n-1} \to X_n \to Y)$$

is in $\langle R^{\perp} \rangle$. Therefore, $I = \langle R^{\perp} \rangle$ and $(k\mathcal{F})^! \cong k\mathcal{Q}^{opp}/\langle R^{\perp} \rangle = k\mathcal{Q}^{opp}/I \cong I(\mathcal{L}^*)$. $\quad \square$

Corollary 1.28. *The* Ext-*algebra of $I(\mathcal{L}^*)$ is isomorphic to the face semigroup algebra $k\mathcal{F}$.*

1.9.4 The Hochschild (Co)Homology of the Face Semigroup Algebra

Let A be a k-algebra and M an A-bimodule. There is a complex of A-bimodules

$$\cdots \xrightarrow{d_{i+1}} M \otimes_k A^{\otimes i} \xrightarrow{d_i} \cdots \xrightarrow{d_1} M \otimes_k A \xrightarrow{d_0} M$$

with maps $d_i : M \otimes_k A^{\otimes i} \to M \otimes A^{\otimes i-1}$ defined by $d_0(m \otimes a) = am - ma$ for $m \in M$, $a \in A$ and for $i \geq 1$

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_i) = (ma_1 \otimes a_2 \otimes \cdots \otimes a_i)$$
$$+ \sum_{j=1}^{i-1} (-1)^j (m \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i)$$
$$+ (-1)^i (a_i m \otimes a_1 \otimes \cdots \otimes a_{i-1}),$$

where $m \in M$ and $a_1, \ldots, a_i \in A$. The *Hochschild homology* of A with coefficients in M is $\mathrm{HH}_i(A, M) = \ker(d_i)/\operatorname{im}(d_{i+1})$ for $i \geq 0$. Let $\mathrm{HH}_i(A) = \mathrm{HH}_i(A, A)$.

Similarly, there exists a cocomplex of A-bimodules

$$M \xrightarrow{d^0} \operatorname{Hom}_k(A, M) \xrightarrow{d^1} \operatorname{Hom}_k(A \otimes_k A, M) \xrightarrow{d^2} \cdots$$

where $d^0 : M \to \operatorname{Hom}_k(A, M)$ is the map $d^0(m)(a) = am - ma$ and d^i is the map $d^i : \operatorname{Hom}_k(A^{\otimes i}, M) \to \operatorname{Hom}_k(A^{\otimes i+1}, M)$ given by

$$(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{i+1})$$
$$+ \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1})$$
$$+ (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1},$$

where $f \in \operatorname{Hom}_k(A^{\otimes i}, M)$ and $a_1, \ldots, a_{i+1} \in A$. The *Hochschild cohomology* of A with coefficients in M is $\mathrm{HH}^i(A, M) = \ker(d^i)/\operatorname{im}(d^{i-1})$ for $i \geq 0$. Denote the Hochschild cohomology of A with coefficients in A by $\mathrm{HH}^i(A) = \mathrm{HH}^i(A, A)$.

Proposition 1.29. *The Hochschild homology* $\mathrm{HH}_i(k\mathcal{F})$ *and the Hochschild cohomology* $\mathrm{HH}^i(k\mathcal{F})$ *of* $k\mathcal{F}$ *vanish in positive degrees. In degree zero the Hochschild homology is* $\mathrm{HH}_0(k\mathcal{F}) \cong k^{\#\mathcal{L}}$ *and the Hochschild cohomology is* $\mathrm{HH}^0(k\mathcal{F}) \cong k$.

Proof. Let $\mathcal{Q}$ denote the quiver of $k\mathcal{F}$. The Hochschild homology of algebras whose quivers have no oriented cycles is known to be zero in positive degrees and k^q in degree 0, where q is the number of vertices in the quiver [Cibils, 1986]. This establishes the Hochschild homology of $k\mathcal{F}$ since $\mathcal{Q}$ has no oriented cycles.

Buchweitz (§3.5 of [Keller, 2003]) proved that the Hochschild cohomology algebra of a Koszul algebra is the Hochschild cohomology algebra of its Koszul dual. Since $k\mathcal{F}$ is a Koszul algebra with Koszul dual the incidence algebra $I(\mathcal{L}^*)$ of the lattice $\mathcal{L}^*$, there is an isomorphism

$$\mathrm{HH}^*(k\mathcal{F}) \cong \mathrm{HH}^*(I(\mathcal{L}^*)) \cong \bigoplus_{i \geq 0} \mathrm{HH}^i(I(\mathcal{L}^*)).$$

Gerstenhaber and Schack ([Gerstenhaber and Schack, 1983]; also see [Cibils, 1989, Corollary 1.4]) proved that the Hochschild cohomology $\mathrm{HH}^i(I(\mathcal{L}^*))$ of $I(\mathcal{L}^*)$ is the simplicial cohomology of the simplicial complex $\Delta(\mathcal{L}^*)$ whose i-simplices are the chains of length i in the poset $\mathcal{L}^*$. Therefore,

$$\mathrm{HH}^i(I(\mathcal{L}^*)) \cong H^i(\Delta(\mathcal{L}^*), k).$$

The latter is zero in positive degrees since $\Delta(\mathcal{L}^*)$ is a double cone ($\mathcal{L}^*$ contains both a top and bottom element) and is k in degree zero since $\Delta(\mathcal{L}^*)$ is connected. It is easy to check directly that $\mathrm{HH}^0(k\mathcal{F}) \cong k$, completing the proof. $\qquad\square$

1.10   Connections with Poset Cohomology

1.10.1   The Cohomology of a Poset

Let P denote a finite poset. The *order complex* $\Delta(P)$ of P is the simplicial complex with i-simplices the chains of length i in P. Suppose P has both a minimal element $\hat{0}$ and a maximal element $\hat{1}$ and let k denote a field. The *order cohomology of P*

with coefficients in k is the reduced simplicial cohomology with coefficients in k of the order complex $\Delta(P - \{\hat{0}, \hat{1}\})$ of $P - \{\hat{0}, \hat{1}\}$. The order cohomology of P has the following characterization in terms of the chains of P.

Suppose P contains at least two distinct elements. For $i \geq 0$, let $C_i(P)$ denote the k-vector space spanned by the i-chains of $P - \{\hat{0}, \hat{1}\}$,

$$C_i(P) = \operatorname{span}_k \left\{ (x_0 < \cdots < x_i) \mid x_j \in P - \{\hat{0}, \hat{1}\} \right\}.$$

For $i = -1$, let $C_{-1}(P) = k$, the vector space spanned by the empty chain. If P consists of one element, then define $C_{-2}(P) = k$ and $C_i(P) = 0$ otherwise.

Define coboundary morphisms $\delta_i : C_i(P) \to C_{i+1}(P)$ by

$$\delta_i(x_0 < \cdots < x_i)$$
$$= \sum_{j=0}^{i+1} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_i),$$

where $x_{-1} = \hat{0}$ and $x_{i+1} = \hat{1}$. It is straightforward to check that $\delta^2 = 0$. The order cohomology of P is $H^i(P) = H^i(P; k) = \ker(\delta_i)/\operatorname{im}(\delta_{i-1})$.

Notice that if P consists of exactly one element, then $H^{-2}(P) = k$ and $H^i(P) = 0$ for $i \neq -2$. If $P = \{\hat{0}, \hat{1}\}$, then $H^{-1}(P) = k$ and $H^i(P) = 0$ for $i \neq -1$.

1.10.2 A Vector Space Decomposition of the Face Semigroup Algebra

Suppose the length of the longest chain in the poset P is $d + 2$. Then $\ker(\delta_d)$ is spanned by the chains of length d in $P - \{\hat{0}, \hat{1}\}$ and $\operatorname{im}(\delta_{d-1})$ is spanned by the elements,

$$\sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_{d-1}), \tag{1.10.1}$$

one for each chain $x_0 \lessdot \cdots \lessdot x_{j-1} < x_j \lessdot \cdots \lessdot x_{d-1}$ of length $d-1$.

Put $P = \mathcal{L}$ in the above and identify the cover relations with the arrows in $\mathcal{Q}$. Then the top cohomology of $\mathcal{L}$ corresponds to the quotient of the span of the maximal paths in $\mathcal{Q}$ by the quiver relations. This gives a vector space isomorphism $e_{\hat{0}} k \mathcal{F} e_{\hat{1}} \cong H^{d-2}(\mathcal{L})$, where the length of the longest chain in $\mathcal{L}$ is d. Folkman [Folkman, 1966] showed that the cohomology of a geometric lattice is non-vanishing only in the top degree. Since $\mathcal{L}^*$ is a geometric lattice and $\Delta(\mathcal{L}^*) = \Delta(\mathcal{L})$, the cohomology of $\mathcal{L}$ is non-vanishing only in the top degree. Therefore, $e_{\hat{0}} k \mathcal{F} e_{\hat{1}} \cong H^*(\mathcal{L})$. Since every interval of a geometric lattice is also a geometric lattice, the result holds for every interval of $\mathcal{L}$. That is, $e_X k \mathcal{F} e_Y \cong H^*([X, Y])$.

Proposition 1.30. *$k\mathcal{F}$ has a k-vector space decomposition in terms of the order cohomology of the intervals of $\mathcal{L}$,*

$$k\mathcal{F} \cong \bigoplus_{X,Y \in \mathcal{L}} H^*([X, Y]).$$

1.10.3 Another Cohomology Construction on Posets

In light of the above decomposition, the direct sum $\bigoplus_{X,Y \in \mathcal{L}} H^*([X, Y])$ inherits a k-algebra structure from $k\mathcal{F}$. This section shows that the algebraic structure can be obtained via the cup product of a cohomology algebra on the intersection lattice. This cohomology construction appears to be new.

Let P be a finite poset and let k denote a field. Let $D_i(P)$ denote the k-vector space of i-chains in P,

$$D_i(P) = \Big\{ (x_0 < \cdots < x_i) \;\Big|\; x_j \in P \Big\}.$$

Define coboundary morphisms $d_i : D_i(P) \to D_{i+1}(P)$ by

$$d_i(x_0 < \cdots < x_i)$$

$$= \sum_{j=1}^{i} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_i).$$

Then $d^2 = 0$. The cohomology groups of the cocomplex $(D_\bullet, d)$ will be denoted by $\mathcal{H}^i(P) = \mathcal{H}^i(P; k) = \ker(d_i)/\operatorname{im}(d_{i-1})$.

The differences between $\mathcal{H}^i(P)$ and $H^i(P)$ are small, but important. The former is defined for any poset P, not just a poset with $\hat{0}$ and $\hat{1}$. The vector space $D_i(P)$ is spanned by *all* the chains in P, not just those avoiding $\hat{0}$ and $\hat{1}$. The summation in the coboundary morphism $d_i : D_i(P) \to D_{i+1}(P)$ runs from $j = 1$ to $j = i$, whereas the summation runs from $j = 0$ to $j = i+1$ in the coboundary morphism $\delta_i : C_i(P) \to C_{i+1}(P)$. However, there is a strong relationship between $\mathcal{H}(P)$ and $H(P)$.

Proposition 1.31. *Let P be a finite poset. Then for all $i \geq 0$,*

$$\mathcal{H}^i(P) \cong \bigoplus_{x,y \in P} H^{i-2}([x,y]).$$

Proof. $D_i(P)$ decomposes into subspaces spanned by the i-chains of P beginning at x and terminating at y: $(x < x_1 < \cdots < x_{i-1} < y)$. The differential d_i respects this decomposition and the subspaces are isomorphic to $C_{i-2}([x,y])$ (drop the x and y of each chain). This isomorphism commutes with the coboundary operators, establishing the proposition. $\square$

The benefit of working with $\mathcal{H}^*(P)$ is that the *simplicial cup product* (see [Munkres, 1984, §49]) on the simplices of the order complex $\Delta(P)$ of P descends to a product on the cohomology.

Define a product $\smile: D_p(P) \times D_q(P) \to D_{p+q}(P)$ by

$$(x_0 < \cdots < x_p) \smile (y_0 < \cdots < y_q)$$

$$
= \begin{cases} (x_0 < \cdots < x_p = y_0 < \cdots < y_q), & x_p = y_0, \\ \\ 0, & x_p \neq y_0. \end{cases} \tag{1.10.2}
$$

Lemma 1.32. *For $c \in D_p(P)$ and $d \in D_q(P)$,*

$$
\delta_{p+q}(c \smile d) \;=\; \delta_p(c) \smile d \;+\; (-1)^p c \smile \delta_q(d).
$$

Proof. Let $c = (x_0 < \cdots < x_p)$ and $d = (x_p < \cdots < x_{p+q})$. Then

$$
\delta_p(c) \smile d \;+\; (-1)^p c \smile \delta_q(d)
$$

$$
= \sum_{j=1}^{p} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_p) \smile d
$$

$$
+ (-1)^p c \smile \sum_{j=p+1}^{p+q} (-1)^{j-p} \sum_{x_{j-1} < x < x_j} (x_p < \cdots < x_{j-1} < x < x_j < \cdots < x_{p+q})
$$

$$
= \sum_{j=1}^{p} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_{p+q})
$$

$$
+ \sum_{j=p+1}^{p+q} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_{p+q})
$$

$$
= \sum_{j=1}^{p+q} (-1)^j \sum_{x_{j-1} < x < x_j} (x_0 < \cdots < x_{j-1} < x < x_j < \cdots < x_{p+q})
$$

$$
= \delta_{p+q}(x_0 < \cdots < x_{p+q})
$$

$$
= \delta_{p+q}(c \smile d). \qquad \square
$$

Corollary 1.33. *The product $D_p(P) \times D_q(P) \xrightarrow{\smile} D_{p+q}(P)$ induces a well-defined product $\mathcal{H}^p(P) \times \mathcal{H}^q(P) \xrightarrow{\smile} \mathcal{H}^{p+q}(P)$ giving $\mathcal{H}^*(P) = \bigoplus_i \mathcal{H}^i(P)$ a k-algebra structure.*

1.10.4 The Face Semigroup Algebra as a Cohomology Algebra

Combining Propositions 1.30 and 1.31 gives the vector space isomorphism

$$\phi : \mathcal{H}^*(\mathcal{L}) \xrightarrow{\cong} \bigoplus_{X,Y \in \mathcal{L}} H^*([X,Y]) \xrightarrow{\cong} k\mathcal{Q}/I \xrightarrow{\varphi} k\mathcal{F}.$$

Recall that Proposition 1.30 identifies $\bigoplus_{X,Y} H^*([X,Y])$ with $k\mathcal{F}$ via the quiver $\mathcal{Q}$ with relations of $k\mathcal{F}$. The isomorphism identifies an unrefinable chain in $\mathcal{L}$ with the corresponding path in $\mathcal{Q}$

$$\left(X_0 \lessdot X_1 \lessdot \cdots \lessdot X_{j-1} \lessdot X_j \right) \longmapsto \left(X_j \to X_{j-1} \to \cdots \to X_1 \to X_0 \right)$$

and maps the relations in $\mathcal{H}^*(\mathcal{L})$ to the quiver relations. Under this isomorphism the multiplication in $\mathcal{H}^*(\mathcal{L})$ maps to the multiplication in $k\mathcal{Q}/I$ (composition of chains in $\mathcal{L}$ maps to composition of paths in $\mathcal{Q}$). Therefore, ϕ is a k-algebra isomorphism.

Proposition 1.34. *Let $k\mathcal{F}$ be the face semigroup algebra of a hyperplane arrangement with intersection lattice $\mathcal{L}$. Then $k\mathcal{F} \cong \mathcal{H}^*(\mathcal{L})$.*

1.10.5 Connection with the Whitney Cohomology of the Intersection Lattice

We finish this section by identifying the Whitney cohomology of $\mathcal{L}$ in $k\mathcal{F}$. (See [Bacławski, 1975] and more recently [Wachs, 1999].) The *Whitney cohomology* of a poset P with $\hat{0}$ is the direct sum $\mathrm{WH}^*(P) = \bigoplus_{X \in P} H^*([\hat{0}, X])$. Since the Whitney homology of $\mathcal{L}^*$ is isomorphic to the Orlik-Solomon algebra of $\mathcal{L}^*$ ([Björner, 1992, §7.10]), the following result also explains how the dual of the Orlik-Solomon algebra embeds in the face semigroup algebra.

Corollary 1.35. *The Whitney cohomology of $\mathcal{L}^*$ is isomorphic to the ideal of chambers in $k\mathcal{F}$. It is a projective indecomposable $k\mathcal{F}$-module.*

Proof. Since $H^*([X,Y]) \cong e_X k\mathcal{F} e_Y$ for all $X, Y \in \mathcal{L}$ (see the discussion preceeding Proposition 1.30), the Whitney cohomology of $\mathcal{L}^*$ is

$$\text{WH}^*(\mathcal{L}^*) \cong \bigoplus_{X \in \mathcal{L}} H^*([X, \hat{1}]) \cong \bigoplus_{X \in \mathcal{L}} e_X k\mathcal{F} e_{\hat{1}} \cong k\mathcal{F} e_{\hat{1}} \cong k\mathcal{F}_{\hat{1}}. \quad \square$$

1.11 Future Directions

These results extend to the semigroup algebra of the semigroup of covectors of an oriented matroid [Björner et al., 1993, §4.1]. This is essential due to two observations. The first observation is that the exact sequence used to construct the projective resolutions of the simple modules (Section 1.7) can be extended to the semigroup algebra of an oriented matroid [Brown and Diaconis, 1998, §6]. The second observation is that the construction of the complete set of primitive orthogonal idempotents in Section 1.5.1 holds for a larger class of semigroups (see Chapter 3).

The cohomology construction introduced in Section 1.10.3 appears to be new and deserves some attention. The questions asked of the order cohomology of a poset P should be asked of $\mathcal{H}^*(P)$. For example, if G is a group acting on a poset P, then this G-action on P induces a G-module structure on $\mathcal{H}^*(P)$, and it would be interesting to study the resulting G-module structure. For order homology and cohomology this has already been extensively studied and is quite interesting. See [Wachs, 1999], for example.

For certain classes of posets $\mathcal{H}^*(P)$ has nice algebraic structure. For example, if P is a Cohen-Macaulay poset, then its incidence algebra $I(P)$ is a Koszul algebra [Polo, 1995, Proposition 1.6]. Hence, $\mathcal{H}^*(P)$ is the Koszul dual algebra of $I(P)$.

This describes the Koszul dual algebra of $I(P)$ in terms of the order cohomology of P.

The construction also provides an extension of a result [Hozo, 1996] describing a part of the Lie algebra (co)homology of a certain subalgebra $N(P)$ of the incidence algebra of P in terms of the order (co)homology of P. Hozo showed that if P contains $\hat{0}$ and $\hat{1}$, then the Lie algebra (co)homology of $N(P)$ contains the order (co)homology of P. His proof extends to show that for any poset P (not necessarily containing $\hat{0}$ and $\hat{1}$), the Lie algebra (co)homology of $N(P)$ contains the (co)homology $\mathcal{H}^*(P)$. This is a further step towards describing the complete Lie algebra (co)homology of $N(P)$ in terms of the combinatorics of the poset P.

1.12 Appendix: Ext^2 and Relations.

This Appendix provides a proof of the following Proposition. Although a reference to a proof of the result is included in the body of this work (see [Bongartz, 1983]), it is included here for the convenience of the reader.

Proposition 1.36. *Let Q be a quiver with no oriented cycles and let I be an admissible ideal. Suppose that R is a minimal set of relations generating I as a two-sided ideal of kQ. Then the number of relations from x to y in R is the dimension of the k-vector space $\text{Ext}^2_{kQ/I}(S_x, S_y)$.*

Proof. Let $R(x, y)$ denote the set of relations in R beginning at x and ending at y. We begin by showing that the number of relations in $R(x, y)$ is the dimension of $y\big(I/(PI + IP)\big)x$.

First note that distinct elements of $R(x, y)$ give distinct elements of $y\big(I/(PI + IP)\big)x$. Suppose $\rho \neq \tau \in R(x, y)$. If $\rho - \tau \in y(PI + IP)x$, then $\rho = \tau + \sum_i a_i p_i \gamma_i + \sum_i b_i \delta_i q_i$, where $a_i, b_i \in k$ are scalars, $p_i \in yQ$ are nonzero paths ending at y, $q_i \in Qx$ are nonzero paths beginning at x, $\gamma_i \in Ix$ are relations beginning at x and $\delta_i \in yI$ are relations ending at y. Note that the lengths of the paths in the linear combinations γ_i and δ_i are strictly less than the lengths of the paths in ρ. Therefore, since R generates I, it follows that we can write γ_i and δ_i in terms of elements of $R - \rho$. Therefore, $\rho \in \langle R - \{\rho\} \rangle$, contradicting that R is a minimal generating set of I. A similar argument shows that the images of the elements in $R(x, y)$ in $yIx/y(PI + IP)x$ are independent. This establishes the inequality $|R(x, y)| \leq \dim y\big(I/(PI + IP)\big)x$. Since R generates I, it follows that $R(x, y) = R \cap yIx$ gives a spanning set for $yIx/y(PI + IP)x$. This establishes the reverse inequality.

We now show that the dimension of $\operatorname{Ext}^2_A(S_x, S_y)$ is the dimension of $y\big(I/(PI + IP)\big)x$, completing the proof. Consider the following sequence of kQ/I-modules.

$$0 \longrightarrow \big(I/IP\big)x \xrightarrow{\ h\ } \big(P/IP\big)x \xrightarrow{\ g\ } \big(kQ/I\big)x \xrightarrow{\ f\ } S_x \longrightarrow 0,$$

where $f(q + I) = q + P$, $g(p + IP) = p + I$ and $h(i + IP) = i + IP$, for $q \in kQ$, $p \in P$ and $i \in I$. This sequence is exact, and $big(P/IP)x$ is a projective kQ/I-module: The map $\bigoplus_{\{y : x \to y \in Q_1\}} kQy \to Px$ given by right multiplication by the arrows $x \to y$ is an isomorphism of kQ-modules, so Px is a projective kQ-module and it follows that $\big(P/IP\big)x$ is a projective kQ/I-module.

Since $\big(P/IP\big)x$ and $\big(kQ/I\big)x$ are projective kQ/I-modules, the above exact sequence allows us to compute $\operatorname{Ext}^2_{kQ/I}(S_x, S_y)$. Proposition 7.2 of Cartan and Eilenberg's *Homological Algebra* gives

$$\operatorname{Ext}^2_{kQ/I}(S_x, S_y) \cong \frac{\operatorname{Hom}_{kQ/I}\big(\big(I/IP\big)x,\ S_y\big)}{h^*\big(\operatorname{Hom}_{kQ/I}\big(\big(P/IP\big)x,\ S_y\big)\big)}.$$

It is straightforward to show the image of h^* is zero. (Any element in I is a linear combination of paths of length at least two. Use the module structure to commute the action of an arrow from these paths onto S_y and note that $P \cdot S_y = 0$.) Therefore,

$$\operatorname{Ext}^2_{kQ/I}(S_x, S_y) \cong \operatorname{Hom}_{kQ/I}((I/IP)x, S_y)$$

$$\cong \operatorname{Hom}_{kQ/P}((I/(PI + IP))x, S_y) \cong \operatorname{Hom}_k(y(I/(PI + IP))x, k).$$

Hence, $\dim \operatorname{Ext}^2_{kQ/I}(S_x, S_y) = \dim(y\big(I/(PI + IP)\big)x).$ $\qquad\square$

1.13 Appendix: Chamber-Valued Derivations of the Face Semigroup Algebra

In this section we show that every derivation $\delta : k\mathcal{F} \to k\mathcal{C}$ taking values in the subspace of $k\mathcal{F}$ spanned by the chambers $\mathcal{C}$ of the arrangement is inner. Equivalently, we show the first Hochscild cohomology of $k\mathcal{F}$ over $k\mathcal{C}$ is trivial. This question arose in early attempts to compute the Hochschild cohomology of $k\mathcal{F}$ over $k\mathcal{F}$.

We begin by recalling some definitions. The first Hochschild cohomology group of $k\mathcal{F}$ over $k\mathcal{C}$ is $\mathrm{HH}^1(k\mathcal{F}, k\mathcal{C}) = \mathrm{Der}(k\mathcal{F}, k\mathcal{C})/\mathrm{Der}^0(k\mathcal{F}, k\mathcal{C})$, where

$$\mathrm{Der}(k\mathcal{F}, k\mathcal{C}) = \left\{ \delta \in \mathrm{Hom}_k(k\mathcal{F}, k\mathcal{C}) \mid \delta(ab) = \delta(a)b + \delta(b)a \text{ for all } a, b \in k\mathcal{F} \right\},$$

$$\mathrm{Der}^0(k\mathcal{F}, k\mathcal{C}) = \left\{ \delta \in \mathrm{Hom}_k(k\mathcal{F}, k\mathcal{C}) \mid \delta(a) = am - ma \text{ for some } m \in k\mathcal{C} \right\}.$$

Proposition 1.37. $\mathrm{HH}^1(k\mathcal{F}, k\mathcal{C}) = 0.$

Proof. Let $\delta : k\mathcal{F} \to k\mathcal{C}$ be a derivation. We show δ is an inner derivation by proceeding as follows.

(a) *We can suppose $\delta(e_X) = 0$ for each $X \in \mathcal{L}$.* Put $m = \sum_{Y \in \mathcal{L}} \delta(e_Y)e_Y$, and $\partial_m(a) = ma - am$ for all $a \in k\mathcal{F}$. Observe that for all $X, Y \in \mathcal{L}$,

$$\delta(e_X e_Y) = e_X \delta(e_Y) + \delta(e_X)e_Y.$$

Thus, for all $X \in \mathcal{L}$,

$$\begin{aligned}
\partial_m(e_X) &= \sum_{Y \in \mathcal{L}} \delta(e_Y)e_Y e_X - \sum_{Y \in \mathcal{L}} e_X \delta(e_Y)e_Y \\
&= \delta(e_X)e_X - \sum_{Y \in \mathcal{L}} \left(\delta(e_X e_Y) - \delta(e_X)e_Y \right) e_Y \\
&= \delta(e_X)e_X - \delta(e_X)e_X + \delta(e_X) \sum_{Y \in \mathcal{L}} e_Y \\
&= \delta(e_X).
\end{aligned}$$

Therefore, $\delta' = \delta - \partial_m$ is a derivation with the property that $\delta'(e_X) = 0$ for all $X \in \mathcal{L}$.

(b) *We can suppose $\delta(c) = 0$ for some chamber c.* Let c be a chamber and construct a complete system of primitive orthogonal idempotents starting from c. That is, suppose $e_{\hat{1}} = c$. Then by (a), $\delta(c) = 0$.

(c) *For any pair of adjacent chambers d, d', there exists $\lambda \in k$ such that $\delta(d - d') = \lambda(d - d')$.* Let d and d' be adjacent chambers with common codimension one face y. Then $\delta(d - d') = \delta(yd - yd') = \delta(y)(d - d') + y\delta(d - d') = y\delta(d - d')$ since $\delta(y) \in k\mathcal{C}$. Thus, $\delta(d - d')$ is a linear combination of chambers adjacent to y. Since y is of codimension 1, there are exactly two chambers adjacent to y. These are d and d'. Thus, $\delta(d - d') = \lambda d + \gamma d'$ for some $\lambda, \gamma \in k$. Then for any chamber c we have $(\lambda + \gamma)c = c\delta(d - d') = \delta(cd - cd') - \delta(c)(d - d') = \delta(0) - \delta(c)(0) = 0$, since $\delta(c) \in k\mathcal{C}$. So $\lambda = -\gamma$. Therefore, for any pair of adjacent chambers d, d', there exists $\lambda \in k$ such that $\delta(d - d') = \lambda(d - d')$.

(d) *Suppose $\mathcal{A}$ is a rank two central arrangement. There exists a $\lambda \in k$ such that $\delta(d - d') = \lambda(d - d')$ for any pair of adjacent chambers d, d'.* Ordering the chambers in clockwise order gives a sequence of pairwise adjacent chambers, $d_0, d_1, \ldots, d_l$. By (c), for each t, there exists $\lambda_t \in k$ such that $\delta(d_{t-1} - d_t) = \lambda_t(d_{t-1} - d_t)$. Note that d_0 and d_l are also adjacent, so $\delta(d_0 - d_l) = \lambda_0(d_0 - d_l)$ for some $\lambda_0 \in k$. Thus,

$$\lambda_0(d_0 - d_l) = \delta(d_0 - d_l) = \delta(d_0 - d_1) + \cdots + \delta(d_{l-1} - d_l)$$

$$= \lambda_1(d_0 - d_1) + \lambda_2(d_1 - d_2) + \cdots + \lambda_l(d_{l-1} - d_l)$$

$$= \lambda_1 d_0 + (\lambda_2 - \lambda_1)d_1 + \cdots + (\lambda_l - \lambda_{l-1})d_{l-1} - \lambda_l d_l.$$

Since chambers are linearly independent, $\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_l$.

(e) *For every $Y \in \mathcal{L}$ there exists $\lambda \in k$ such that $\delta(d - d') = \lambda(d - d')$ for*

chambers d and d' with common codimension one face of support Y. If $\text{supp}(y') = \text{supp}(y) = Y$, then $y'd$ and $y'd'$ are two distinct adjacent chambers. So (c) gives the existence of $\lambda' \in k$ such that $\delta(y'd - y'd') = \lambda'(y'd - y'd')$. Similarly, there exists $\lambda \in k$ such that $\delta(d - d') = \lambda(d - d')$. Since δ is a derivation, $\lambda'(y'd - y'd') = \delta(y'd - y'd') = \delta(y')(d - d') + y'\delta(d - d') = y'\delta(d - d') = \lambda(y'd - y'd')$. Therefore, $\lambda = \lambda'$.

(f) *There exists $\lambda \in k$ such that for any pair of adjacent chambers d, d', $\delta(d - d') = \lambda(d - d')$.* Let d, d' and e, e' denote two pairs of adjacent chambers with common codimension one faces y and w, respectively. By (c) there exists $\lambda \in k$ such that $\delta(d - d') = \lambda(d - d')$ and $\lambda' \in k$ such that $\delta(e - e') = \lambda'(e - e')$. Since $\text{supp}(w)$ and $\text{supp}(y)$ are hyperplanes and $\mathcal{A}$ is a central arrangement, $\text{supp}(w)$ and $\text{supp}(y)$ intersect in a codimension two subspace Z. Let z be an element with $\text{supp}(z) = Z$. Then the chambers zd, zd' are adjacent and distinct with common codimension one face zy of support $\text{supp}(zy) = \text{supp}(z) \wedge \text{supp}(y) = \text{supp}(y)$. Thus, $\delta(zd - zd') = \lambda(zd - zd')$ by (e). Similarly, $\delta(ze - ze') = \lambda'(ze - ze')$. The restricted arrangement $\mathcal{A}_{\geq Z}$ is a rank two central arrangement, so by (d), $\lambda = \lambda'$.

(g) *For any chamber d, $\delta(d) = \lambda(d - c)$ for some $\lambda \in k$.* Pick a gallery $d, d_1, \ldots, d_l, c$. Then by (f),

$$\delta(d) = \delta(d - d_1) + \delta(d_1 - d_2) + \cdots \delta(d_l - c)$$
$$= \lambda(d - d_1) + \lambda(d_1 - d_2) + \cdots \lambda(d_l - c)$$
$$= \lambda(d - c).$$

(h) *δ is an inner derivation.* It is the inner derivation $\partial_{\lambda c}$: for any $x \in \mathcal{F}$,

$$\delta(x) = \delta(x)c = \delta(xc) - x\delta(c) = \delta(xc) = \lambda(xc - c) = x(\lambda c) - (\lambda c)x = \partial_{\lambda c}(x),$$

where we used the facts that $\delta(x) \in k\mathcal{C}$, $\delta(c) = 0$, $xc \in \mathcal{C}$ and (g). Therefore, δ is an inner derivation. $\qquad \square$

Corollary 1.38. *Let $M = \mathrm{rad}(k\mathcal{C}) \cong (1 - e_{\hat{1}})k\mathcal{C}$. Then $\mathrm{HH}^1(k\mathcal{F}, M) = k$.*

Proof. Let $\delta \in \mathrm{Der}(k\mathcal{F}, M)$. Since $M \subset k\mathcal{C}$, $\delta \in \mathrm{Der}(k\mathcal{F}, k\mathcal{C})$. Therefore, $\delta \in \mathrm{Der}^0(k\mathcal{F}, k\mathcal{C})$ since $\mathrm{HH}^1(k\mathcal{F}, k\mathcal{C}) = 0$. It follows from the above proof that $\delta = \partial_s$ where $s = \sum_{X \in \mathcal{L}} \delta(e_X)e_X + \lambda e_{\hat{1}} = \delta(e_{\hat{1}}) + \lambda e_{\hat{1}}$. Hence, $\delta = \partial_{\delta(e_{\hat{1}})} + \lambda \partial_{e_{\hat{1}}}$. Since $\delta(e_{\hat{1}}) \in M$, $\partial_{\delta(e_{\hat{1}})} \in \mathrm{Der}(k\mathcal{F}, M)$. Thus, $[\delta] = [\partial_{e_{\hat{1}}}]$. It remains to show $\partial_{e_{\hat{1}}} \notin \mathrm{Der}^0(k\mathcal{F}, M)$.

Suppose $\partial_{e_{\hat{1}}} = \partial_m$ for some nonzero $m \in M$. Then, $\partial_{e_{\hat{1}}}(a) = \partial_m(a)$ for every $a \in k\mathcal{F}$. Hence, $e_{\hat{1}}a - ae_{\hat{1}} = ma - am$ for every $a \in k\mathcal{F}$. Take $a = e_{\hat{1}}$. Then $0 = e_{\hat{1}}e_{\hat{1}} - e_{\hat{1}}e_{\hat{1}} = me_{\hat{1}} - e_{\hat{1}}m = me_{\hat{1}} = m$. Here we used that $m \in M = (1 - e_{\hat{1}})k\mathcal{C}$ and that $m \in k\mathcal{C}$, respectively. $\qquad \square$

CHAPTER 2

THE FACE SEMIGROUP ALGEBRA OF A REFLECTION ARRANGEMENT

2.1 Introduction

This chapter investigates the face semigroup algebra of hyperplane arrangements that arise from finite reflection groups. There is a hyperplane arrangement associated to every finite reflection group consisting of the hyperplanes that are fixed by some reflection in the reflection group. This leads to an action of the finite reflection group on the face semigroup algebra of the arrangement. Patrick Bidigare [Bidigare, 1997] showed that the subalgebra of elements invariant under this group action is anti-isomorphic to the descent algebra of the finite reflection group. The latter is a subalgebra of the group algebra of the finite reflection group [Solomon, 1976]. Bidigare's result introduces a new setting in which the descent algebra can be studied. Manfred Schocker [Schocker, 2004, Schocker, 2005] has initiated this study by studying the descent algebra of the symmetric group from this point of view.

The material presented here is the first installment of an ongoing project to study the module structure of the descent algebra of an arbitrary finite reflection group. In the following the quiver of the descent algebra of the symmetric group is computed. Altough this was already known — it is implicit in [Garsia and Reutenauer, 1989] and stated explictly in [Schocker, 2004] — the approach presented here is sufficiently abstract that it may generalize to all finite reflection groups. (The quiver of the descent algebra of any other finite reflection group is not known.) There is only one piece missing. If Lemma 2.15 holds for

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any finite reflection group, then the proofs presented here carry over verbatim to the general setting. Otherwise, an understanding of why the Lemma fails to hold should provide enough information to determine the quiver.

The first few sections present some definitions and recall some of the theory. Section 2.2 recalls the definition of a finite reflection group W and the reflection arrangement associated to W. Section 2.3 presents the definition of the descent algebra $\mathcal{D}(W)$ of an arbitrary finite reflection group W. In Section 2.4 we recall the anti-isomorphism between $\mathcal{D}(W)$ and the invariant subalgebra $(k\mathcal{F})^W$ of the face semigroup algebra $k\mathcal{F}$ of the reflection arrangement.

The next few sections study some of the module structure of the invariant subalgebra $(k\mathcal{F})^W$. Section 2.5 describes the simple $(k\mathcal{F})^W$-modules. In Section 2.6 a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$ is constructed using a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ and the action of W on $k\mathcal{F}$. This construction requires that the characteristic of the field k does not divide the order of the reflection group W. The idempotents are used to describe the indecomposable projective $(k\mathcal{F})^W$-modules in Section 2.7.

The remaining sections deal with trying to compute the quiver of $(k\mathcal{F})^W$. In Section 2.8 an action of W on the path algebra $k\mathcal{Q}$ of the quiver $\mathcal{Q}$ of $k\mathcal{F}$ is defined, and a W-equivariant surjection $k\mathcal{Q} \to k\mathcal{F}$ is constructed. In Section 2.9 a quiver $\mathcal{Q}^W$ is constructed from $\mathcal{Q}$ using the action of W and a morphism ξ from $k(\mathcal{Q}^W)$ onto $(k\mathcal{F})^W$ is defined. This implies that $\mathcal{Q}^W$ is the quiver of a subalgebra of $(k\mathcal{F})^W$. When $W = S_n$, we show that ξ is surjective, hence $\mathcal{Q}^{S_n}$ is the quiver of $(k\mathcal{F})^{S_n}$. Sections 2.11 and 2.12 study the case $W = S_n$ in more detail. The former presents a combinatorial description of the quiver $\mathcal{Q}^{S_n}$ and the latter presents detailed examples for S_n when $n \leq 6$.

There is a final section 2.13 outlining possible future directions for this project.

2.2 Coxeter Groups and Reflection Arrangements

This section will recall definitions and basic facts from the theory of finite reflection groups. See [Brown, 1989], [Humphreys, 1990] and [Björner and Brenti, 2005] for more information.

A *finite Coxeter group* W (or a *finite reflection group*) is a finite group generated by a set of reflections in a real vector space V. The *reflection arrangement* $\mathcal{A}(W)$ of W is the hyperplane arrangement consisting of the hyperplanes fixed by some reflection in W. The Coxeter group W permutes the hyperplanes in $\mathcal{A}(W)$, so W acts on the intersection lattice $\mathcal{L}(W)$ of $\mathcal{A}(W)$ and on the faces $\mathcal{F}(W)$ of $\mathcal{A}(W)$. The action of W on $\mathcal{F}(W)$ extends linearly to an action of W on the semigroup algebra $k\mathcal{F}(W)$. When the Coxeter group W is clear from the context we will write $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{A}$ for $\mathcal{F}(W)$, $\mathcal{L}(W)$ and $\mathcal{A}(W)$, respectively.

Proposition 2.1. *Let W be a finite reflection group and let $\mathcal{F}$ denote the face semigroup of the reflection arrangement $\mathcal{A}$ of W. Let $\mathcal{L}$ denote the intersection lattice of $\mathcal{A}$.*

1. *$X \leq Y$ iff $w(X) \leq w(Y)$ for all $X, Y \in \mathcal{L}$ and all $w \in W$.*

2. *$w(X \vee Y) = w(X) \vee w(Y)$ for all $X, Y \in \mathcal{L}$ and all $w \in W$.*

3. *$x \leq y$ iff $w(x) \leq w(y)$ for all $x, y \in \mathcal{F}$ and all $w \in W$.*

4. *$w(xy) = w(x)w(y)$ for all $x, y \in \mathcal{F}$ and all $w \in W$.*

5. *$w(\mathrm{supp}(X)) = \mathrm{supp}(wX)$ for all $X \in \mathcal{L}$ and all $w \in W$.*

(1) and (3) say that W acts on the posets $\mathcal{L}$ and $\mathcal{F}$ by poset automorphisms. (2) and (4) say that W acts on the semigroups $\mathcal{L}$ and $\mathcal{F}$ by semigroup automorphisms. (5) says that supp : $\mathcal{F} \to \mathcal{L}$ is W-equivariant. It follows that W acts on $k\mathcal{F}$ and $k\mathcal{L}$ by algebra automorphisms and that supp : $k\mathcal{F} \to k\mathcal{L}$ is a W-equivariant algebra morphism. The proof of the Proposition is straightforward.

For $x \in \mathcal{F}$ let $[x] = \{w(x) : w \in W\}$ denote the W-orbit of x. Let $W_x = \{w \in W : w(x) = x\}$ denote the stabilizer of x, and let W^x denote a set of coset representatives of W_x. Similarly, for $X \in \mathcal{L}$ let $[X] = \{w(X) : w \in W\}$ denote the W-orbit of X, let $W_X = \{w \in W : w(X) = X\}$ denote the subgroup of elements of w that map the subspace X to itself (not necessarily pointwise), and let W^X denote a set of coset representatives for W_X.

Let $\mathcal{L}/W = \{[X] : X \in \mathcal{L}\}$ denote the set of W-orbits of elements in $\mathcal{L}$. Then $\mathcal{L}/W$ is a poset with partial order given by $[X] \leq [Y]$ iff there exists $w \in W$ with $w(X) \leq Y$. (Reflexivity and transitivity of $\leq$ are straightforward; to see that $\leq$ is anti-symmetric note that $w(X) \leq X$ iff $w(X) = X$ for all $w \in W$, $X \in \mathcal{L}$.) We will denote elements of $\mathcal{L}/W$ by $[X]$ where $X \in \mathcal{L}$ is a representative for that orbit. If $[X] \leq [Y]$ in $\mathcal{L}/W$, we can suppose, without loss of generality that $X \leq Y$ in $\mathcal{L}$.

2.3   The Descent Algebra of a Finite Coxeter Group

Let W denote a finite Coxeter group and let c denote a chamber in the reflection arrangement $\mathcal{A}$ of W. If $x \leq c$ is a codimension one face of c, then the hyperplane supp(x) is called a *wall* of c. Let $S \subset W$ denote the set of reflections in the walls of c. Then S is a generating set of W [Brown, 1989, §I.5A] and the pair (W, S) is called a *Coxeter system*.

Fix a Coxeter system (W, S). For $J \subset S$ let $W_J = \langle J \rangle$ denote the subgroup of W

generated by the elements in J. Each coset of W_J in W contains a unique element of minimal length when expressed as a word in the generators S [Humphreys, 1990, Proposition 1.10(c)]. Let W^J denote the set of these *minimal length coset representatives*. Let $w_J = \sum_{w \in W^J} w$ denote the formal sum of the minimal length coset representatives of W_J. Then w_J is an element of the group algebra kW of W with coefficients in some field k. The k-vector space $\mathcal{D}(W)$ spanned by the elements w_J for $J \subset S$ is a subalgebra of the group algebra kW called the *descent algebra* of W. It was introduced by Solomon [Solomon, 1976].

2.4   The Descent Algebra as the Invariant Subalgebra

This section recalls the anti-isomorphism between the descent algebra of a finite Coxeter group W and the W-invariant subalgebra of the face semigroup algebra $k\mathcal{F}$ of the reflection arrangement of W.

Let W denote a finite Coxeter group, let c be a chamber in the reflection arrangement $\mathcal{A}$ of W and let S denote the set of reflections in the walls of c as in the previous section. Let $\mathcal{C}$ denote the set of chambers in the reflection arrangement $\mathcal{A}$. The k-vector space $k\mathcal{C}$ spanned by the chambers $\mathcal{C}$ is a two-sided ideal of the face semigroup algebra $k\mathcal{F}$ of $\mathcal{A}$. Therefore, it is a $k\mathcal{F}$-module and hence a $(k\mathcal{F})^W$-module, where $(k\mathcal{F})^W$ denotes the subalgebra of $k\mathcal{F}$ consisting of elements invariant under the action of W. The action of W on $k\mathcal{C}$ commutes with the action of $(k\mathcal{F})^W$ on $k\mathcal{C}$, so there is an algebra morphism $(k\mathcal{F})^W \to \mathrm{End}_{kW}(k\mathcal{C})$, where $\mathrm{End}_{kW}(k\mathcal{C})$ denotes the k-algebra of kW-endomorphisms of $k\mathcal{C}$. There is an isomorphism $k\mathcal{C} \cong kW$ of kW-modules given by identifying $w(c)$ with w for all $w \in W$. This gives a k-algebra isomorphism $\mathrm{End}_{kW}(k\mathcal{C}) \cong \mathrm{End}_{kW}(kW)$. Since any kW-endomorphism commuting with the action of W is given by right multiplication by

an element of kW, there is an isomorphism $\text{End}_{kW}(kW) \cong (kW)^{op}$, where $(kW)^{op}$ is the k-algebra obtained from kW be reversing the multiplication in kW. The last statement of the following result was first noticed in [Bidigare, 1997].

Theorem 2.2 ([Brown, 2000]). *Let W be any finite Coxeter group. The following composition is injective with image the descent algebra $\mathcal{D}(W)$ of W.*

$$(k\mathcal{F}(W))^W \hookrightarrow \text{End}_{kW}(k\mathcal{C}) \cong \text{End}_{kW}(kW) \cong \text{End}_{kW}(kW) \cong (kW)^{op}.$$

Therefore, $(k\mathcal{F}(W))^W$ is anti-isomorphic to $\mathcal{D}(W)$.

Before proceeding we explicitly record the above isomorphism. Recall that any face $x \in \mathcal{F}$ of the reflection arrangement is in the W-orbit of a unique face y of c [Humphreys, 1990, Theorem 1.12]; and that the stabilizer of $y \leq c$ is W_J, where J is the set of reflections in the walls of c containing y [Humphreys, 1990, Theorem 1.15]. Therefore, the elements $\sum_{w \in W^J} w(y)$ for $y \leq c$ form a basis of $(k\mathcal{F})^W$. The above anti-isomorphism is given by sending $\sum_{w \in W^J} w(y)$ to $\sum_{w \in W^J} w$.

2.5 Simple Modules

We now describe the simple $(k\mathcal{F})^W$-modules. Since the support map supp $: k\mathcal{F} \to k\mathcal{L}$ is W-equivariant it restricts to an algebra morphism supp $: (k\mathcal{F})^W \to (k\mathcal{L})^W$ from the W-invariant subalgebra $(k\mathcal{F})^W$ of $k\mathcal{F}$ to the W-invariant subalgebra $(k\mathcal{L})^W$ of $k\mathcal{L}$. The semigroup algebra $k\mathcal{L}$ is isomorphic to the ring $k^{\mathcal{L}}$ of linear functions from $\mathcal{L}$ to k [Solomon, 1967]. The isomorphism is given by $\phi : k\mathcal{L} \to k^{\mathcal{L}}$ mapping X to the function that is 1 on $Y \geq X$ and that is 0 otherwise. The left action of W on $\mathcal{L}$ induces a right action of W on $k^{\mathcal{L}}$ given by $(fw)(X) = f(w(X))$ for $f \in k^{\mathcal{L}}$, $w \in W$ and $X \in \mathcal{L}$. Under these W-actions ϕ is W-equivariant

and restricts to an isomorphism $(k\mathcal{L})^W \cong (k^{\mathcal{L}})^W$. Now $(k^{\mathcal{L}})^W \cong k^{\mathcal{L}/W}$ via the isomorphism $\rho(f)([X]) = f(X)$ with inverse $\rho^{-1}(g)(X) = g([X])$, where $X \in \mathcal{L}$, $[X] \in \mathcal{L}/W$ is the W-orbit of X, $f \in (k^{\mathcal{L}})^W$ and $g \in k^{\mathcal{L}/W}$. Therefore, there is a surjective algebra morphism with a nilpotent kernel,

$$(k\mathcal{F})^W \xrightarrow{\text{supp}} (k\mathcal{L})^W \xrightarrow{\phi} (k^{\mathcal{L}})^W \xrightarrow{\rho} k^{\mathcal{L}/W}.$$

The components of this morphism give the irreducible characters of $(k\mathcal{F})^W$.

Proposition 2.3. *The irreducible representations of $(k\mathcal{F})^W$ are 1-dimensional. There is one irreducible representation for each orbit $[X] \in \mathcal{L}/W$, given by the restriction of $\chi_X : k\mathcal{F} \to k$ to $(k\mathcal{F})^W$.*

Recall that χ_X is the irreducible character (§1.4.3) of $k\mathcal{F}$ defined on $y \in \mathcal{F}$ by

$$\chi_X(y) = \begin{cases} 1, & \text{if } \text{supp}(y) \leq X, \\ 0, & \text{otherwise.} \end{cases}$$

If $y \in \mathcal{F}$ and W^y denotes a set of coset representatives for the stabilizer W_y of y, then

$$\chi_X \left(\sum_{w \in W^y} w(y) \right) = \sum_{w \in W^y} \chi_X\left(w(y)\right) = |\{w \in W^y : w(\text{supp}(y)) \leq X\}|.$$

If $\text{supp}(x)$ is in the orbit of X, then we can suppose without loss of generality that $\text{supp}(x) = X$. Then for any $w \in W$, $w(X) \leq X$ iff $w(X) = X$. Therefore,

$$\chi_X \left(\sum_{w \in W^x} w(x) \right) = |\{w \in W^x : w(X) = X\}| = [W_X : W_x],$$

where W_X is the subgroup of W of elements w such that $w(X) = X$.

2.6 Primitive Idempotents

This section constructs a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$. The construction requires that the characteristic of the field k does

not divide the order of W. We use the construction from Section 1.5 to obtain a complete system of primitive orthogonal idempotents $\{e_X\}_{X \in \mathcal{L}}$ in $k\mathcal{F}$ that behave well under the action of W. Specifically, we will have $w(e_X) = e_{w(X)}$ for all $X \in \mathcal{L}$ and $w \in W$. Then the elements $\sum_{Y \in [X]} e_Y$, one for each orbit $[X] \in \mathcal{L}/W$, give a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$. Indeed, they are are invariant under the action of W; they are idempotents since the sum of orthogonal idempotents is an idempotent; they are orthogonal because the summands are orthogonal; they sum to one since $\sum_X e_X = 1$; and they are primitive idempotents since they lift primitive idempotents in $k^{\mathcal{L}/W}$:

$$(\rho \circ \phi \circ \mathrm{supp})\left(\sum_{w \in W^X} e_{w(X)}\right)([Y]) = \left(\sum_{w \in W^X} (\rho \circ \phi \circ \mathrm{supp})(e_{w(X)})\right)([Y])$$

$$= \left(\sum_{w \in W^X} \rho(\delta_{w(X)})\right)([Y])$$

$$= \sum_{w \in W^X} \delta_{w(X)}(Y)$$

$$= \delta_{[X]}([Y]),$$

where W^X is a set of coset representatives of W_X. In the above, δ is the Krönecker δ-function.

For each $X \in \mathcal{L}$ let $\widehat{X}$ denote a linear combination of elements of support X whose coefficients sum to 1. Suppose that $w(\widehat{X}) = \widehat{w(X)}$ for all $w \in W$ and all $X \in \mathcal{L}$ (see below for examples of such elements). Construct a complete system of primitive orthogonal idempotents $\{e_X\}_{X \in \mathcal{L}}$ in $k\mathcal{F}$ using the formula $e_X = \widehat{X} - \sum_{Y > X} \widehat{X} e_Y$ (see Remark 1.6). That $w(e_X) = e_{w(X)}$ for all $w \in W$ follows by induction on $X \in \mathcal{L}$. Indeed, if $w \in W$, and $X = \hat{1}$, then $w(e_{\hat{1}}) = w(\widehat{1}) = \widehat{1} = e_{\hat{1}} =$

$e_{w(\hat{1})}$. Now suppose that $w(e_Y) = e_{w(Y)}$ for all $Y > X$. Then

$$w(e_X) = w\widehat{X} - \sum_{Y > X} w\left(\widehat{X}e_Y\right)$$

$$= w\widehat{X} - \sum_{Y > X} w\left(\widehat{X}\right)w(e_Y)$$

$$= \widehat{wX} - \sum_{Y > X} \widehat{wX}e_{wY}$$

$$= \widehat{wX} - \sum_{Y > wX} \widehat{wX}e_Y$$

$$= e_{wX}.$$

The above is summarized in the following statement.

Proposition 2.4. *For each $X \in \mathcal{L}$ let $\widehat{X}$ denote a linear combination of elements of support X whose coefficients sum to 1. Suppose that $w(\widehat{X}) = \widehat{w(X)}$ for all $w \in W$ and $X \in \mathcal{L}$. Define e_X for $X \in \mathcal{L}$ recursively by $e_X = \widehat{X} - \sum_{Y > X} \widehat{X}e_Y$. Then the elements $\sum_{Y \in [X]} e_Y$, one for each orbit $[X] \in \mathcal{L}/W$, form a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$.*

Example 2.5. For each $X \in \mathcal{L}$, let $\widetilde{X}$ denote the normalized sum of all elements of support X.

$$\widetilde{X} = \frac{1}{\#\{x \in \mathcal{F} : \mathrm{supp}(x) = X\}}\left(\sum_{\mathrm{supp}(x) = X} x\right).$$

Then $w(\widetilde{X}) = \widetilde{w(X)}$ for all $w \in W$ and so the idempotents

$$e_X = \widetilde{X} - \sum_{Y > X} \widetilde{X}e_Y$$

satisfy $w(e_X) = e_{w(X)}$ for all $X \in \mathcal{L}$ and the elements

$$\sum_{Y \in [X]} e_Y \text{ for } [X] \in \mathcal{L}/W$$

form a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$.

Example 2.6. Fix a chamber c in $\mathcal{F}$. Define a relation on the faces of c by $x \sim y$ iff $\text{supp}(x) = w(\text{supp}(y))$ for some $w \in W$. Then $\sim$ is an equivalence relation and we let Ψ denote a set of representatives for the equivalence relation $\sim$. Since every face $x \in \mathcal{F}$ is W-conjugate to a unique face of c [Humphreys, 1990, Theorem 1.12], it follows that every $X \in \mathcal{L}$ is W-conjugate to the support of a unique element in Ψ. That is, for every $X \in \mathcal{L}$ there is a $w \in W$ and a unique $y \in \Psi$ such that $X = w(\text{supp}(y))$. Let $x = w(y)$ and let x^+ denote the normalized sum of elements of support $X = \text{supp}(x)$ in the W-orbit of x,

$$ x^+ = \frac{1}{|\{y \in [x] : \text{supp}(y) = X\}|} \left(\sum_{\substack{y \in [x] \\ \text{supp}(y) = X}} y \right). $$

(Note that $|\{y \in [x] : \text{supp}(y) = X\}|$ is the index of W_x in W_X, so it is nonzero in k since the characteristic of k does not divide the order of W.) Since x^+ is a weighted sum of elements in the orbit of x, it does not depend on the choice of $w \in W$ above. Therefore, the element x^+ depends only on the support of x and we denote x^+ by X^+, where $X = \text{supp}(x)$. The following Lemma shows that $w(X^+) = w(X)^+$. Therefore, we obtain a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$ by appealing to the above Proposition.

Lemma 2.7. *If $w \in W$ and $X \in \mathcal{L}$, then $w(X^+) = w(X)^+$.*

Proof. Let $x, y \in \mathcal{F}$ and let $X = \text{supp}(x)$. If y is in the orbit of x and $\text{supp}(y) = X$, then $w(y)$ is in the orbit of $w(x)$ and $\text{supp}(w(y)) = w(X)$. This gives a bijection between $\{y \in \mathcal{F} : y \in [x], \text{supp}(y) = X\}$ and $\{z \in \mathcal{F} : z \in [w(x)], \text{supp}(z) = w(X)\}$. Hence, $w(x^+) = (w(x))^+$ for any $x \in \mathcal{F}$.

Let $X \in \mathcal{L}$ and $w \in W$. There exists a unique $y \in \Psi$ such that $X = \text{supp}(u(y))$ for some $u \in W$. So $X^+ = (u(y))^+$ by definition. If $Z = wX$, then $Z = (wu)\,\text{supp}(y)$. Therefore, $(w(X))^+ = Z^+ = ((wu)(y))^+ = w(u(y)^+) = w(X^+)$. $\quad\square$

2.7 Projective Indecomposable Modules

In this section we use the idempotents constructed in the previous section to describe the indecomposable projective $(k\mathcal{F})^W$-modules.

Let $\{e_X\}_{X \in \mathcal{L}}$ denote the complete system of primitive orthogonal idempotents in $k\mathcal{F}$ constructed in Example 2.6. Then $e_X = X^+ - \sum_{Y > X} X^+ e_Y$ for all $X \in \mathcal{L}$, where X^+ was defined by

$$X^+ = \frac{1}{[W_X : W_x]} \left(\sum_{w \in W^x \cap W^X} w(x) \right) = \frac{1}{|\{y \in [x] : \mathrm{supp}(y) = X\}|} \left(\sum_{\substack{y \in [x] \\ \mathrm{supp}(y) = X}} y \right),$$

where x has support X and is W-conjugate to an element in Ψ. Then the set $\left\{ \sum_{w \in W^X} e_{w(X)} \right\}_{X \in \mathcal{L}}$ is a complete system of primitive orthogonal idempotents for $(k\mathcal{F})^W$ (Proposition 2.4). This affords a $(k\mathcal{F})^W$-module decomposition

$$(k\mathcal{F})^W \cong \bigoplus_{[X] \in \mathcal{L}/W} (k\mathcal{F})^W \left(\sum_{w \in W^X} e_{w(X)} \right)$$

and the $(k\mathcal{F})^W$-modules $(k\mathcal{F})^W \left(\sum_{w \in W^X} e_{w(X)} \right)$ are all the projective indecomposable $(k\mathcal{F})^W$-modules, upto isomorphism.

Proposition 2.8. *Let $X \in \mathcal{L}$. Then*

$$(k\mathcal{F})^W \left(\sum_{w \in W^X} e_{w(X)} \right) = \mathrm{span} \left\{ \left(\sum_{w \in W^x} w(x) \right) \left(\sum_{u \in W^X} e_{u(X)} \right) : [\mathrm{supp}(x)] = [X] \right\}.$$

Therefore, $\mathrm{rad}(k\mathcal{F})^W$ is spanned by differences $\sum_{w \in W^x} w(x) - \sum_{u \in W^y} u(y)$ with $[\mathrm{supp}(x)] = [\mathrm{supp}(y)]$.

Proof. If $x \in \mathcal{F}$ with $\mathrm{supp}(x) = v(X)$ for some $v \in W$, then

$$\left(\sum_{w \in W^x} w(x) \right) \left(\sum_{u \in W^X} e_{u(X)} \right) = \sum_{u \in W^X} \left(\sum_{w \in W^x} w(x) \right) e_{u(X)}$$

$$= \sum_{u \in W^X} \left(\sum_{Y \in [X]} \sum_{\substack{w \in W^x \\ w(\mathrm{supp}(x)) = Y}} w(x) \right) e_{u(X)}$$

$$= \sum_{u \in W^X} \left(\sum_{\substack{w \in W^x \\ w(\text{supp}(x))=u(X)}} w(x) \right) e_{u(X)}$$

$$= \sum_{u \in W^X} \left(\frac{|W_X|}{|W_x|} u(X^+) \right) e_{u(X)}$$

$$= \frac{|W_X|}{|W_x|} \sum_{u \in W^X} e_{u(X)}.$$

Therefore, $\frac{|W_x|}{|W_X|} \left(\sum_{w \in W^x} w(x) \right) \left(\sum_{u \in W^X} e_{u(X)} \right) = \sum_{u \in W^X} e_{u(X)}$. It follows that the $(k\mathcal{F})^W$-module $(k\mathcal{F})^W \left(\sum_{w \in W^X} e_{w(X)} \right)$ is spanned by elements of the form

$$\left(\sum_{v \in W^y} v(y) \right) \left(\sum_{u \in W^X} e_{u(X)} \right) = \left(\sum_{v \in W^y} v(y) \right) \left(\sum_{w \in W^x} w(x) \right) \left(\sum_{u \in W^X} e_{u(X)} \right).$$

Since the support of $v(y)w(x)$ is $v(\text{supp}(y)) \vee w(\text{supp}(x)) \geq w(\text{supp}(x))$, the product $\left(\sum_{v \in W^y} v(y) \right) \left(\sum_{w \in W^x} w(x) \right)$ is a linear combination of elements of the form $\left(\sum_{w \in W^z} w(z) \right)$ with $[\text{supp}(z)] \geq [\text{supp}(x)]$. But if $[\text{supp}(z)] \not\leq [X]$, then $\left(\sum_{w \in W^z} w(z) \right) \left(\sum_{u \in W^X} e_{u(X)} \right) = 0$ (by Lemma 1.4). Therefore, $[\text{supp}(z)] = [X]$.

$\square$

2.8   A W-Equivariant Quiver Map

This section lifts the W-action on $\mathcal{F}$ to a W-action on $k\mathcal{Q}$ and constructs a quiver map $\varphi : k\mathcal{Q} \to k\mathcal{F}$ that is W-equivariant.

We begin by recalling the construction of the map $\varphi : k\mathcal{Q} \to k\mathcal{F}$, where $\mathcal{F}$ is the face semigroup of an arbitrary hyperplane arrangement and $\mathcal{Q}$ is the quiver of $k\mathcal{F}$. Then the construction will be specialized to reflection arrangements using the primitive idempotents constructed above. This will allow us to define an action of W on $k\mathcal{Q}$ and we will show that under this W-action the map $\varphi : k\mathcal{Q} \to k\mathcal{F}$ is W-equivariant.

Fix an orientation ϵ_X on each subspace $X \in \mathcal{L}$. For $x, y \in \mathcal{F}$ with $x \lessdot y$, define numbers $[x : y]$ by

$$[x : y] = \epsilon_X(\vec{x}_1, \ldots, \vec{x}_t)\epsilon_Y(\vec{x}_1, \ldots, \vec{x}_t, \vec{y}_1),$$

where $\vec{x}_1, \ldots, \vec{x}_t$ is a basis of X and $\vec{y}_1$ is a vector in y. Define a map $\partial : k\mathcal{F} \to k\mathcal{F}$ on $x \in \mathcal{F}$ by

$$\partial(x) = \sum_{y > x}[x : y]y.$$

For each $X \in \mathcal{L}$ let L_X denote a nonempty set of elements of support X and let $\lambda_X = |L_X|$. Then the elements $\{e_X\}_{X \in \mathcal{L}}$ constructed using the recursion

$$e_X = \left(\frac{1}{\lambda_X} \sum_{x \in L_X} x\right) - \left(\frac{1}{\lambda_X} \sum_{x \in L_X} x\right)\left(\sum_{Y > X} e_Y\right)$$

is a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ (see Remark 1.6). There is a k-algebra morphism $\varphi : k\mathcal{Q} \to k\mathcal{F}$ defined by

$$\varphi(X) = e_X \text{ for each vertex } X \in \mathcal{Q},$$

$$\varphi(Y \to X) = \lambda_X\left(e_X \partial(x) e_Y\right) \text{ for each arrow } (X \to Y) \in \mathcal{Q},$$

where x is any face of support X. The kernel of φ is generated by the sum of all the paths of length two in $\mathcal{Q}$ (see Section 1.8.3B).

Now let $\mathcal{A}$ be the reflection arrangement of the finite Coxeter group W and suppose that $\{e_X\}_{X \in \mathcal{L}}$ is one of the complete systems of primitive orthogonal idempotents defined in Section 2.6. Define $\varphi : k\mathcal{Q} \to k\mathcal{F}$ as above using these idempotents. Define an action of W on $k\mathcal{Q}$ as follows. For $w \in W$ and $X \in \mathcal{L}$, let

$$\sigma_X(w) = \epsilon_X(\vec{x}_1, \ldots, \vec{x}_s)\epsilon_{wX}(w\vec{x}_1, \ldots, w\vec{x}_s),$$

where $\vec{x}_1, \ldots, \vec{x}_s$ is a basis of the subspace X. For $w \in W$ and a path $(X_0 \to \cdots \to X_t)$ in $\mathcal{Q}$ define

$$w(X_0 \to \cdots \to X_t) = \sigma_{X_0}(w)\sigma_{X_t}(w)(w(X_0) \to \cdots \to w(X_t)).$$

Here X_i is viewed as a subspace of the ambient vector space V and $w(X_i)$ is defined using the action of w on V. For any $w \in W$ and any vertex X in $\mathcal{Q}$ we have $wX = \sigma_X(w)^2 w(X) = w(X)$. If $(X_0 \to \cdots \to X_t)$ is a path in $\mathcal{Q}$, then

$$w(X_0 \to \cdots \to X_t) = w(X_{t-1} \to X_t) \cdots w(X_0 \to X_1)$$

since $\sigma_X(w)^2 = 1$ for all $X \in \mathcal{L}$. Therefore $w : k\mathcal{Q} \to k\mathcal{Q}$ is an algebra isomorphism.

The following gives a geometric interpretation of $\sigma_X(w)\sigma_Y(w)$ if $Y \lessdot X$.

Lemma 2.9. *Suppose $Y \lessdot X$ in $\mathcal{L}$ and let $w \in W$ with $w(X) = X$ and $w(Y) = Y$. Then $\sigma_X(w)\sigma_Y(w) = -1$ iff w swaps the halfspaces of X determined by Y.*

Proof. If $w(X) = X$, then $\sigma_X(w) = \epsilon_X(\vec{x}_1, \ldots, \vec{x}_s)\epsilon_{w(X)}(w\vec{x}_1, \ldots, w\vec{x}_s)$ is the sign of the transformation $w|_X$. So if M is the matrix of $w|_X$ then $\sigma_X(w) = \det(M)$.

Now suppose that $Y \lessdot X$ and $w(X) = X$ and $w(Y) = Y$. Let $\vec{x}_1, \ldots, \vec{x}_s$ be a positively oriented orthonormal basis for Y. Then for some $\vec{v} \in X$ orthogonal to Y, the vectors $\vec{x}_1, \ldots, \vec{x}_s, \vec{v}$ form a positively oriented orthonormal basis for X. Since w is an orthogonal transformation, $w(\vec{v})$ is orthogonal to Y since $\vec{v}$ is orthogonal to Y and $w(\vec{v}_1), \ldots, w(\vec{x}_s)$ is an orthonormal basis for Y. Therefore, $w(\vec{v}) = \lambda\vec{v}$, where $\lambda \in \{\pm 1\}$. Let M be the matrix of $w|_Y$ with respect to the basis $\vec{x}_1, \ldots, \vec{x}_s$. Then the matrix N of $w|_X$ with respect to the basis $\vec{x}_1, \ldots, \vec{x}_s, \vec{v}$ is

$$N = \begin{bmatrix} M & 0 \\ 0 & \lambda \end{bmatrix}.$$

Therefore, $\sigma_X(w) = \det(N) = \lambda\det(M) = \lambda\sigma_Y(w)$. Hence, $\sigma_X(w)\sigma_Y(w) = \lambda$. Therefore, $\sigma_X(w)\sigma_Y(w)$ is -1 if w swaps the halfspaces of X determined by Y and is +1 otherwise. $\square$

The following describes the behaviour of the incidence numbers under the action of W on $\mathcal{F}$.

Lemma 2.10. *Suppose $x \lessdot y$ in $\mathcal{F}$. If $w \in W$, then*

$$[wx : wy] = \sigma_{\mathrm{supp}(x)}(w)\sigma_{\mathrm{supp}(y)}(w)[x : y].$$

Proof. (For a geometric proof use the proceeding lemma.) Let $X = \mathrm{supp}(x)$ and $Y = \mathrm{supp}(y)$. Let $x_1, \ldots, x_t$ be a *positively oriented* basis of X and v_y be a vector in y. Then $wx_1, \ldots, wx_t$ is a basis of wX and wv_y is a vector in wy. Therefore,

$$\sigma_X(w)\sigma_Y(w)[x : y] = \epsilon_{wX}(wx_1, \ldots, wx_t)\epsilon_{wY}(wx_1, \ldots, wx_t, wv_y)\epsilon_Y(x_1, \ldots, x_t, v_y)^2$$

$$= \epsilon_{wX}(wx_1, \ldots, wx_t)\epsilon_{wY}(wx_1, \ldots, wx_t, wv_y)$$

$$= [wx : wy]. \qquad \square$$

We now describe the action of W on the images of the arrows of $\mathcal{Q}$.

Lemma 2.11. *Suppose $X \lessdot Y$. If $\mathrm{supp}(x) = X$ and $w \in W$, then*

$$w(\lambda_X e_X \partial(x) e_Y) = \sigma_X(w)\sigma_Y(w)\ \lambda_{w(X)} e_{w(X)}\partial(wx)e_{w(Y)}.$$

Proof. Note that

$$\partial(x)e_Y = \left(\sum_{y > x}[x : y]y\right)e_Y = \left(\sum_{\substack{y > x \\ \mathrm{supp}(y) = Y}}[x : y]y\right)e_Y$$

since $ye_Y = 0$ if $\mathrm{supp}(y) \not\leq Y$ (Lemma 1.4). Since $X \lessdot Y$ there are exactly two faces y, y' with support Y having x as a codimension one face [Brown, 1989, §I.4E Proposition 3]. Therefore

$$\partial(x)e_Y = ([x : y]y + [x : y']y')\ e_Y.$$

For $w \in W$, Lemma 2.10 and our assumption that $w(e_X) = e_{w(X)}$ for all $X \in \mathcal{L}$, give

$$w(e_X\partial(y)e_Y) = w\left(e_X\left([x : y]y + [x : y']y'\right)e_Y\right)$$

$$= w(e_X)w\Big([x:y]y + [x:y']y'\Big)w(e_Y)$$

$$= e_{w(X)}\Big([x:y]wy + [x:y']wy'\Big)e_{w(Y)}$$

$$= \sigma_X(w)\sigma_Y(w)\Big(e_{w(X)}\Big([wx:wy]wy + [wx:wy']wy'\Big)e_{w(Y)}\Big)$$

$$= \sigma_X(w)\sigma_Y(w)\ e_{w(X)}\partial(wx)e_{w(Y)}.$$

The number λ_X depends only on the orbit of X, so the result follows. $\qquad\square$

We can now prove that φ is W-equivariant.

Proposition 2.12. $\varphi : k\mathcal{Q} \to k\mathcal{F}$ *is W-equivariant.*

Proof. Let $w \in W$. If X is a vertex in $\mathcal{Q}$, then $w(\varphi(X)) = w(e_X) = e_{w(X)} = \varphi(w(X))$. If $Y \to X$ is an arrow in $\mathcal{Q}$, then Lemma 2.11 gives

$$w\left(\varphi(Y \to X)\right) = w\left(\lambda_X e_X \partial(x)e_Y\right)$$

$$= \sigma_X(w)\sigma_Y(w)\ \Big(\lambda_{w(X)}e_{wX}\partial(wx)e_{wY}\Big)$$

$$= \sigma_X(w)\sigma_Y(w)\ \varphi(wY \to wX)$$

$$= \varphi(w(Y \to X)).$$

If $X_0 \to \cdots \to X_s$ is a path in Q, then

$$w\left(\varphi(X_0 \to \cdots \to X_s)\right) = w\left(\varphi\Big((X_{s-1} \to X_s)\cdots(X_0 \to X_1)\Big)\right)$$

$$= w\Big(\varphi(X_{s-1} \to X_s)\cdots\varphi(X_0 \to X_1)\Big)$$

$$= w\Big(\varphi(X_{s-1} \to X_s)\Big)\cdots w\Big(\varphi(X_0 \to X_1)\Big)$$

$$= \varphi\Big(w(X_{s-1} \to X_s)\Big)\cdots\varphi\Big(w(X_0 \to X_1)\Big)$$

$$= \varphi\Big(w(X_{s-1} \to X_s)\cdots w(X_0 \to X_1)\Big)$$

$$= \varphi\Big(w(X_0 \to \cdots X_s)\Big),$$

where we used the facts that $\varphi : k\mathcal{Q} \to k\mathcal{F}$ and $w : k\mathcal{F} \to k\mathcal{F}$ are algebra morphisms and that $w\left(\varphi(X \to Y)\right) = \varphi(w(X \to Y))$. $\qquad\square$

2.9   The Quiver of the Invariant Subalgebra

In this section a quiver $\mathcal{Q}^W$ will be defined using the action of W on $\mathcal{Q}$. It will then be shown that $\mathcal{Q}^W$ is the quiver of a *subalgebra* of $(k\mathcal{F})^W$. When $W = S_n$, this subalgebra is precisely $(k\mathcal{F})^{S_n}$, giving that $\mathcal{Q}^{S_n}$ is the quiver of $(k\mathcal{F})^{S_n}$. The quiver of the descent algebra $\mathcal{D}(S_n)$ is obtained by reversing the arrows of $\mathcal{Q}^{S_n}$.

Let $\mathcal{Q}^W$ denote the quiver on the vertex set $\mathcal{L}/W$ defined as follows. Suppose $[Y] \lessdot [X]$ in $\mathcal{L}/W$. By replacing Y with $w(Y)$ for a suitable $w \in W$ we can suppose $Y \lessdot X$ in $\mathcal{L}$. There is an arrow $[X] \to [Y]$ in $\mathcal{Q}^W$ iff there does not exist a $w \in W$ satisfying $w(X \to Y) = -(X \to Y)$. Equivalently, there is no $w \in W_X \cap W_Y$ with $\sigma_X(w)\sigma_Y(w) = -1$. Geometrically, this condition states that there is no $w \in W$ that swaps the two halfspaces of X determined by Y (see Lemma 2.9).

Define a map $\psi : k(\mathcal{Q}^W) \to k\mathcal{Q}$ by

$$\psi([X]) = \sum_{Y \in [X]} Y \qquad \text{and} \qquad \psi([X] \to [Y]) = \sum_{w \in W^{X \to Y}} w(X \to Y),$$

for vertices $[X]$ in $\mathcal{Q}^W$ and arrows $[X] \to [Y]$ in $\mathcal{Q}^W$, where $W^{X \to Y}$ denotes a set of coset representatives of the stabilizer $W_{X \to Y}$ of the arrow $X \to Y$.

Lemma 2.13. $\psi : k(\mathcal{Q}^W) \to k\mathcal{Q}$ *is a well-defined k-algebra morphism.*

Proof. To ensure that ψ extends to a k-algebra morphism we need only check that $\psi([Y])\psi([X] \to [Y])\psi([X]) = \psi([X] \to [Y])$ [Assem et al., 2006, §II Theorem 1.8]. Well,

$$\psi([X] \to [Y])\psi([X]) = \sum_{w \in W^{X \to Y}} w(X \to Y) \sum_{r \in W^X} r(X)$$

$$= \sum_{w \in W^{X \to Y}} \left(w(X \to Y) \sum_{r \in W^X} (wr)(X) \right)$$

$$= \sum_{w \in W^{X \to Y}} w \left(\sum_{r \in W^X} (X \to Y)r(X) \right)$$

$$= \sum_{w \in W^{X \to Y}} w \left((X \to Y)X \right)$$

$$= \sum_{w \in W^{X \to Y}} w \left(X \to Y \right)$$

$$= \psi([X] \to [Y]).$$

Similarly, $\psi([Y])\psi([X] \to [Y]) = \psi([X] \to [Y])$. $\qquad\qquad\qquad\square$

Consider the composition $\xi : k(\mathcal{Q}^W) \xrightarrow{\psi} k\mathcal{Q} \xrightarrow{\varphi} k\mathcal{F}$. Since the image of ψ is contained in the W-invariant subalgebra of $k\mathcal{Q}$ and since φ is W-equivariant, the image of ξ is contained in $(k\mathcal{F})^W$. Recall that an ideal of a path algebra is *admissible* if it is contained in the square of the ideal generated by the arrows. That is, every element in the ideal is a linear combination of paths of length at least two.

Lemma 2.14. *The kernel of ξ is an admissible ideal of $k(\mathcal{Q}^W)$.*

Proof. Suppose $\sum_\rho \lambda_\rho \rho \in \ker(\xi)$ where ρ is a path in $\mathcal{Q}^W$. Then $\sum_\rho \lambda_\rho \psi(\rho)$ is a linear combination of paths in kQ with $\varphi\left(\sum_\rho \lambda_\rho \psi(\rho) \right) = 0$ since $\xi = \varphi \circ \psi$. Since $\ker(\varphi)$ is an admissible ideal of $k\mathcal{Q}$, it follows that $\sum_\rho \lambda_\rho \psi(\rho)$ is a linear combination of paths in $\mathcal{Q}$ of length at least two. Therefore, by the definition of ψ, each ρ must have been a path of $\mathcal{Q}^W$ of length at least two. $\qquad\square$

Theorem 1.9(d) of [Auslander et al., 1995] gives a condition to identify the quiver of a finite dimensional k-algebra: Q is the quiver of a finite dimensional split basic k-algebra A iff $A \cong kQ/I$ where I an admissible ideal of kQ. The above

Lemma shows that $\ker(\xi)$ is an admissible ideal of $k(\mathcal{Q}^W)$, therefore $\mathcal{Q}^W$ is the quiver of a subalgebra of $(k\mathcal{F})^W$.

We now proceed to show that when $W = S_n$, this subalgebra is the full algebra $(k\mathcal{F})^W$. We will need the case $p = 2$ of the following result. The general result for all integers $p \geq 0$ will then follow once we know the quiver of $(k\mathcal{F})^W$.

Lemma 2.15. *If $W = S_n$, then for all $p \in \mathbb{N}$,*

$$\mathrm{rad}^p\left((k\mathcal{F})^W\right) = \mathrm{rad}^p\left(k\mathcal{F}\right) \cap (k\mathcal{F})^W.$$

Proof. This is Theorem 9.10 in [Schocker, 2005]. $\square$

Lemma 2.16. *If $W = S_n$, then $\xi : k(\mathcal{Q}^W) \to (k\mathcal{F})^W$ is surjective.*

Proof. We will show that $\xi(k(\mathcal{Q}^W)) + \mathrm{rad}^2\left((k\mathcal{F})^W\right) = (k\mathcal{F})^W$. The result then follows from standard ring theory: if A is a k-algebra and A' is a subalgebra of A such that $A' + \mathrm{rad}^2(A) = A$, then $A' = A$ [Benson, 1998, Proposition 1.2.8].

Since $\varphi : k\mathcal{Q} \to k\mathcal{F}$ is surjective and the vertices and arrows of $\mathcal{Q}$ generate $k\mathcal{Q}$ as a k-algebra, the images of the vertices and arrows of $\mathcal{Q}$ under φ generate $k\mathcal{F}$ as a k-algebra. It follows that $\mathrm{rad}^2(k\mathcal{F})$ is spanned by elements $\varphi(\rho)$ where ρ is a path in $\mathcal{Q}$ of length at least two. Therefore, any element in $(k\mathcal{F})^W$ is a linear combination of elements $\sum_{w \in W^\rho} w(\varphi(\rho))$ where $\rho \in \mathcal{Q}$, and $\sum_{w \in W^\rho} w(\varphi(\rho)) \in \left(\mathrm{rad}^2\left(k\mathcal{F}\right) \cap (k\mathcal{F})^W\right)$ if the length of the path ρ is at least two. By Lemma 2.15, $\sum_{w \in W^\rho} w(\varphi(\rho)) \in \mathrm{rad}^2\left((k\mathcal{F})^W\right)$ if ρ has length at least two. We need only show that the elements $\sum_{w \in W^\rho} w(\varphi(\rho)) \in \xi(k(\mathcal{Q}^W))$ if the length of ρ is at most one. This follows from the fact that φ is W-equivariant:

$$\sum_{w \in W^X} w(\varphi(X)) = \sum_{w \in W^X} e_{w(X)} = \xi([X]),$$

$$\sum_{w \in W^{X \to Y}} w(\varphi(X \to Y)) = \xi([X] \to [Y]). \qquad \square$$

Theorem 2.17. $\mathcal{Q}^{S_n}$ *is the quiver of* $(k\mathcal{F})^{S_n}$. *Therefore, the quiver of the descent algebra of* S_n *is the quiver obtained from* $\mathcal{Q}^{S_n}$ *by reversing the arrows.*

Note that the proof of Lemma 2.16 does not use anything specific about the group S_n except that $\mathrm{rad}^2\left((k\mathcal{F})^{S_n}\right) = \mathrm{rad}^2\left(k\mathcal{F}\right) \cap (k\mathcal{F})^{S_n}$. Therefore, if this holds holds for any finite Coxeter group W, then the quiver of $(k\mathcal{F})^W$ is $\mathcal{Q}^W$. And from this it would follow that $\mathrm{rad}^p\left((k\mathcal{F})^W\right) = \mathrm{rad}^p\left(k\mathcal{F}\right) \cap (k\mathcal{F})^W$ holds for any finite Coxeter group W and for any $p \in \mathbb{N}$. Indeed, if $a \in \mathrm{rad}^p(k\mathcal{F}) \cap (k\mathcal{F})^W$, then since ξ is surjective there is an element $c \in k(\mathcal{Q}^W)$ such that $\xi(c) = a$. Write $c = c_0 + c_1$ where c_0 is a linear combination of paths of length at least p and c_1 is a linear combination of paths of length less than p. Then $a = \xi(c_0) + \xi(c_1)$. It follows that $\xi(c_1)$ must be zero since ξ is grade-preserving and $a \in \bigoplus_{q \geq p}(k\mathcal{F})_q$ (since $a \in \mathrm{rad}^p(k\mathcal{F})$). Since $c_0 \in \mathrm{rad}^p(k(\mathcal{Q}^W))$, we have $a = \xi(c_0) \in \xi\left(\mathrm{rad}^p(k(\mathcal{Q}^W))\right) \subset \mathrm{rad}^p((k\mathcal{F})^W)$. The reverse containment is immediate.

2.10 Quiver Relations

From the proof of Lemma 2.14 it follows that any element in the kernel of ξ is mapped by ψ to an element of $\ker(\varphi)$. Therefore, the quiver relations for $(k\mathcal{F})^W$ are obtained by lifting the relations in $\ker(\varphi)$ to $\mathcal{Q}^W$ using ψ. Recall that the kernel of φ is generated by the sum of all the paths of length two in $[Y, X]$, where $[Y, X]$ is an interval of length two in $\mathcal{Q}$.

It would be very useful to have a description of the relations directly in terms of paths in $\mathcal{Q}^W$.

2.11 The Symmetric Group

This section gives a combinatorial description of the quiver of the algebra $(k\mathcal{F})^W$ for $W = S_n$.

THE REFLECTION ARRANGEMENT. Fix $n \in \mathbb{N}$. The *braid arrangement* is the hyperplane arrangement $\mathcal{A}$ in $V = \mathbb{R}^n$ consisting of the hyperplanes $H_{ij} = \{v \in V : v_i = v_j\}$ for $1 \leq i < j \leq n$. The group of transformations generated by the reflections in the hyperplanes in $\mathcal{A}$ can be identified with the symmetric group S_n acting on V by permuting coordinates: for $\omega \in S_n$ and $v \in V$, define

$$\omega(v) = \omega((v_1, \ldots, v_n)) = (v_{\omega^{-1}(1)}, \ldots, v_{\omega^{-1}(n)}).$$

THE INTERSECTION LATTICE. A *set partition* of $[n] = \{1, \ldots, n\}$ is a collection of nonempty subsets $B = \{B_1, \ldots, B_r\}$ of $[n]$ such that $\bigcup_i B_i = [n]$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. The sets B_i in B are called *blocks*. The elements of the intersection lattice $\mathcal{L}$ of $\mathcal{A}$ are identified with set partitions of $[n]$ via the following,

$$\{B_1, \ldots, B_r\} \leftrightarrow \left\{v \in V : v_i = v_j \text{ if } \exists h \text{ such that } i, j \in B_h\right\} = \bigcap_{h=1}^{r} \left(\bigcap_{i,j \in B_h} H_{ij}\right),$$

where $\{B_1, \ldots, B_r\}$ is a set partition of $[n]$. Under this identification, if B and C are set partitions of $[n]$, then $B \lessdot C$ iff B is obtained from C by merging two blocks of C, and the action of S_n on $\mathcal{L}$ is given by $\omega(\{B_1, \ldots, B_r\}) = \{\omega(B_1), \ldots, \omega(B_r)\}$.

To simplify notation we will concatenate the elements of block. For example, we will write $\{5, 134, 2\}$ instead of $\{\{5\}, \{1, 3, 4\}, \{2\}\}$.

THE POSET OF FACES. Let $v \in V$ be a vector in a chamber of $\mathcal{A}$. Then v is not on any of the hyperplanes H_{ij}, so all the coordinates of v are distinct. Therefore, there exists $\omega \in S_n$ such that $v_{\omega(1)} < \ldots < v_{\omega(n)}$. All vectors in the chamber satisfy this identity, so the chamber can be identified with the permutation ω of $[n]$. The

faces of the chamber are obtained by changing some of the inequalities to equalities, so the faces $\mathcal{F}$ of $\mathcal{A}$ can be identified with *set compositions* (ordered set partitions) of $[n]$. For example, $(5, 134, 2) \leftrightarrow \{v \in V : v_5 < v_1 = v_3 = v_4 < v_2\}$. The partial order on set compositions is $(B_1, \ldots, B_m) \leq (C_1, \ldots, C_l)$ iff $(C_1, \ldots, C_l)$ consists of a set composition of B_1, followed by a set composition of B_2, and so forth. The support map $\mathrm{supp} : \mathcal{F} \to \mathcal{L}$ forgets the order of the blocks in the set composition. The action of S_n on $\mathcal{F}$ is given by $\omega((B_1, \ldots, B_r)) = (\omega(B_1), \ldots, \omega(B_r))$.

THE SEMIGROUP OF FACES. If $(B_1, \ldots, B_l)$ and $(C_1, \ldots, C_m)$ are set partitions of $[n]$, their product is the set partition of $[n]$ given by

$$(B_1, \ldots, B_l)\,(C_1, \ldots, C_m)$$
$$= (B_1 \cap C_1, B_2 \cap C_1, \ldots, B_l \cap C_1, \cdots, B_1 \cap C_m, B_2 \cap C_m, \ldots, B_l \cap C_m)^{\bowtie},$$

where $\bowtie$ means "delete empty intersections".

ORBITS OF SET PARTITIONS. The S_n-orbit of a set partition $\{B_1, \ldots, B_l\}$ of $[n]$ depends only on the sizes of the blocks B_i, so $\mathcal{L}/S_n$ can be identified with the poset of integer partitions of n and the quotient map $\mathcal{L} \to \mathcal{L}/S_n$ is given by $\{B_1, \ldots, B_l\} \mapsto \{|B_1|, \ldots, |B_l|\}$. (Recall that an *integer partition* of n is a set of positive integers whose sum is n.) If $b = \{b_1, \ldots, b_l\}$ and $c = \{c_1, \ldots, c_m\}$ are integer partitions of n, then $b \lessdot c$ iff b is obtained from c by adding two elements of c. That is, $b = \{c_1, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots, c_m\} \cup \{c_i + c_j\}$.

THE QUIVER OF $(k\mathcal{F})^{S_n}$. We can now describe the quiver of $(k\mathcal{F})^{S_n}$.

Theorem 2.18. *The quiver of $(k\mathcal{F})^{S_n}$ is the directed graph with one vertex for each integer partition of n and exactly one arrow $c \to b$ iff b is obtained from c by adding two distinct elements of c.*

Proof. The quiver of $(k\mathcal{F})^{S_n}$ is $\mathcal{Q}^{S_n}$, so we determine $\mathcal{Q}^{S_n}$. Recall that $\mathcal{Q}^{S_n}$ is a

directed graph on the vertex set $\mathcal{L}/S_n$, which we identified with the set of integer partitions of n. Suppose $b = \{b_1, \ldots, b_l\}$ and $c = \{c_1, \ldots, c_m\}$ are integer partitions of n. By the definition of $\mathcal{Q}^{S_n}$, if c does not cover b, then there is no arrow from c to b. So suppose $b \lessdot c$. Let $C = \{C_1, \ldots, C_m\}$ denote a set partition of $[n]$ with $|C_i| = c_i$ for all $1 \leq i \leq m$. Since b is obtained from c by adding two distinct elements of c, the set partition B obtained by merging the corresponding blocks of C satisfies $|B_i| = b_i$ for all $1 \leq i \leq m-1$. By re-indexing B and C we can suppose that $B = \{C_1 \cup C_2, C_3, \ldots, C_m\}$. There is an arrow from c to b in $\mathcal{Q}^{S_n}$ iff there does not exist an $\omega \in S_n$ with $\omega(C \to B) = -(C \to B)$.

Suppose $c_1 = c_2$. Then C_1 and C_2 have the same cardinality. Let ω be a permutation of $[n]$ that swaps these blocks and is the identity on elements not in $C_1 \cup C_2$. Then ω fixes the set composition $(C_1 \cup C_2, C_3, \ldots, C_m)$, so it fixes the subspace B pointwise. (For any Coxeter group W, if an element of W fixes a face set-wise then it fixes the face pointwise [Humphreys, 1990, Proposition 1.15].) And ω maps C to C, but not pointwise since it does not fix the set composition $(C_1, \ldots, C_m)$. Therefore, ω interchanges the halfspaces of C determined by B. It follows from Lemma 2.9 that $\omega(C \to B) = -(C \to B)$. Therefore, there is no arrow from c to b in $\mathcal{Q}^{S_n}$ if b is obtained from c by adding two identical elements of c.

Suppose $c_1 \neq c_2$. If $\omega \in S_n$ with $\omega(B) = B$ and $\omega(C) = C$, then ω permutes the blocks of B and the blocks of C. It follows that $\omega(C_1) = C_1$ and $\omega(C_2) = C_2$. Let $x = (C_1, C_2, C_3, \ldots, C_m)$ and $y = (C_1 \cup C_2, C_3, \ldots, C_m)$. Then, $y\omega(x) = (C_1, C_2, C_3, \ldots, C_m) = x$. So $w(x)$ and x are on the same side of $\text{supp}(y) = B$. It follows from Lemma 2.9 that $\omega(C \to B) = (C \to B)$. Therefore, there is an arrow from c to b in $\mathcal{Q}^{S_n}$ if b is obtained from c by adding two distinct elements of c. $\quad\square$

Corollary 2.19. *The quiver of the descent algebra $\mathcal{D}(S_n)$ of S_n is the quiver*

obtained from $\mathcal{Q}^{S_n}$ by reversing all the arrows.

2.12 Examples: S_n for $n \leq 6$

This section describes in detail the quiver with relations of $(k\mathcal{F})^{S_n}$ for $n \leq 6$.

2.12.1 Computing σ_X for S_n.

Before proceeding to the examples, we record a method for computing the values of the functions σ_X for the symmetric group.

Suppose $B = \{B_1, \ldots, B_m\} \in \mathcal{L}$ is a set partition of $[n]$. By re-indexing the blocks B_i of B, we suppose that B satisfies $\min(B_1) < \cdots < \min(B_m)$. The partition B corresponds to the subspace V_B of vectors $\vec{v} \in V$ satisfying $v_i = v_j$ if $i, j \in B_k$ for some $1 \leq k \leq m$. Therefore, the vectors $\vec{v}_{B_i}$ satisfying $(\vec{v}_{B_i})_j = 1$ if $j \in B_i$ and 0 otherwise is a basis of V_B. Orient V_B by declaring $(\vec{v}_{B_1}, \cdots, \vec{v}_{B_m})$ to be a positive basis. Call this basis the *canonical basis* of V_B. Since the action of S_n on V permutes the coordinates of V, every $\omega \in S_n$ maps the canonical basis of V_B to a permutation of the canonical basis of $V_{\omega(B)}$. This permutation of the canonical basis of $V_{\omega(B)}$ gives a permutation of the blocks of $w(B)$, and $\sigma_B(w)$ is just the sign of this permutation.

Lemma 2.20. *Suppose $B = \{B_1, \ldots, B_m\}$ is a set partition of $[n]$ satisfying $\min(B_1) < \cdots < \min(B_m)$. Then $\sigma_B(w)$ is the sign of the permutation $\tau \in S_m$ satisfying $\min(w(B_{\tau(1)})) < \cdots < \min(w(B_{\tau(m)}))$.*

Proof. Let $\vec{v}_{B_1}, \ldots, \vec{v}_{B_m}$ be the canonical basis for V_B, as above. Therefore, $\sigma_B(\omega) = \epsilon_{w(B)}\left(\omega(\vec{v}_{B_1}), \ldots, \omega(\vec{v}_{B_m})\right)$. This is the determinant of the change of basis matrix mapping $\omega(\vec{v}_{B_1}), \ldots, \omega(\vec{v}_{B_m})$ to the canonical basis of $V_{\omega(B)}$. Since the basis
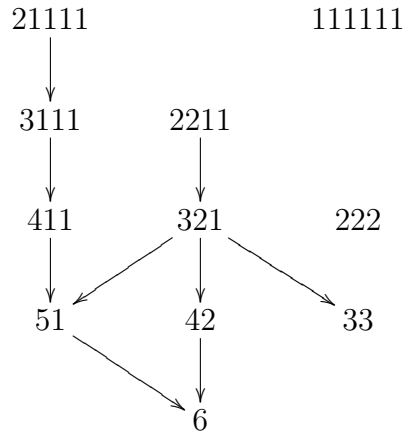
21111   111111

3111   2211

411   321   222

51   42   33

6

FIGURE 2.1: *The quiver $\mathcal{Q}^{S_6}$ of the algebra $(k\mathcal{F})^{S_6}$.*

$\omega(\vec{v}_{B_1}), \ldots, \omega(\vec{v}_{B_m})$ is a permutation of the canonical basis for $V_{\omega(B)}$, the change of basis matrix is a permutation matrix (corresponding to τ), and its determinant is precisely the sign of the permutation. $\qquad\qquad\qquad\qquad\qquad\qquad\square$
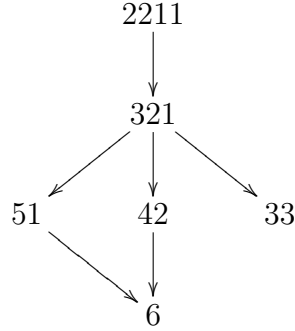
2.12.2   Examples: S_n for $n \leq 5$

It is straightforward to check that the dimension of $k(\mathcal{Q}^{S_n})$ is the dimension of $(k\mathcal{F})^{S_n}$ for $1 \leq n \leq 5$. Therefore, the algebras $k(\mathcal{Q}^{S_n})$ and $(k\mathcal{F})^{S_n}$ are isomorphic for $1 \leq n \leq 5$.
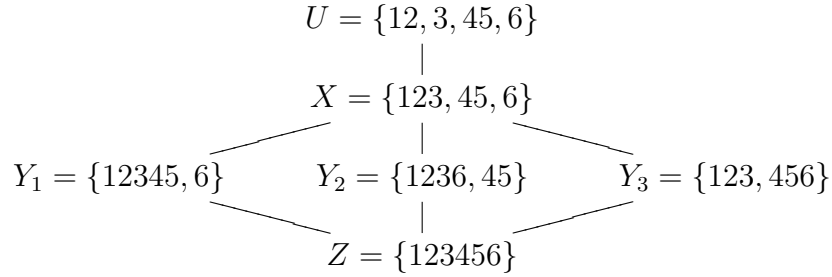
2.12.3   Example: S_6

See Figure 2.1 for the quiver $\mathcal{Q}^{S_6}$. The dimension of $k(\mathcal{Q}^{S_6})$ is 33 while the dimension of $(k\mathcal{F})^{S_6}$ is $2^5 = 32$. Therefore, the kernel of ξ is generated by exactly one nonzero quiver relation. We will show that the relation is given by the sum of the

two paths of length three starting from 2211 quiver of $\mathcal{Q}^{S_6}$.

$$
\begin{array}{c}
2211 \\
\downarrow \\
321 \\
\swarrow \quad \downarrow \quad \searrow \\
51 \qquad 42 \qquad 33 \\
\searrow \quad \downarrow \\
6
\end{array}
$$

Pick representatives in $\mathcal{L}$ of the above orbits.

$$
U = \{12, 3, 45, 6\}
$$
$$
|
$$
$$
X = \{123, 45, 6\}
$$

$$
Y_1 = \{12345, 6\} \qquad Y_2 = \{1236, 45\} \qquad Y_3 = \{123, 456\}
$$

$$
Z = \{123456\}
$$

Then,

$$
\psi(2211 \to 321 \to 42 \to 6)
$$

$$
= \psi(42 \to 6)\, \psi(321 \to 42)\, \psi(2211 \to 321)
$$

$$
= \left(\sum_{w \in S_6^{Y_2 \to Z}} w(Y_2 \to Z) \right) \left(\sum_{u \in S_6^{X \to Y_2}} u(X \to Y_2) \right) \left(\sum_{v \in S_6^{U \to X}} v(U \to X) \right).
$$

If $u(X) \neq v(X)$ or $u(Y_2) \neq w(Y_2)$, then $w(Y_2 \to Z)u(X \to Y_2)v(U \to X) = 0$. So suppose $u(X) = v(X)$ and $u(Y_2) = w(Y_2)$. Since $v(X) = u(X)$ and all the blocks of X are distinct, we have $v(123) = u(123)$, $v(45) = u(45)$ and $v(6) = u(6)$. It follows that $\sigma_X(v) = \sigma_X(u)$ by Lemma 2.20. Since there is only one element in the interval $[Z, v(X)]$ in the orbit of Y_2, it follows that $v(Y_2) = w(Y_2) = u(Y_2)$ since all three are in this interval. It follows that $\sigma_{Y_2}(v) = \sigma_{Y_2}(w) = \sigma_{Y_2}(u)$ by Lemma 2.20. Together these give $u(X \to Y_2) = v(X \to Y_2)$, $w(Y_2 \to Z) = v(Y_2 \to Z)$ and

that $v \in (S_6)_{(U \to X)}$ iff $v \in (S_6)_{(U \to X \to Y_2 \to Z)}$. Therefore,

$$\psi(2211 \to 321 \to 42 \to 6) = \sum_{v \in (S_6)^{(U \to X \to Y_2 \to Z)}} v(U \to X \to Y_2 \to Z).$$

A similar argument gives

$$\psi(2211 \to 321 \to 51 \to 6) = \sum_{v \in (S_6)^{(U \to X \to Y_1 \to Z)}} v(U \to X \to Y_1 \to Z).$$

Combining these with the quiver relations of $\mathcal{Q}$ gives,

$$\left(\xi(2211 \to 321 \to 42 \to 6) + \xi(2211 \to 321 \to 51 \to 6) \right)$$

$$= \left(\sum_{v \in (S_6)^{(U \to X \to Y_3 \to Z)}} v(U \to X \to Y_3 \to Z) \right) + I.$$

The result follows once it is shown that $\sum v(U \to X \to Y_3 \to Z) \in I$. Consider

$$U = \{12, 3, 45, 6\}$$

$$X = \{123, 45, 6\} \qquad\qquad X' = \{12, 3, 456\}$$
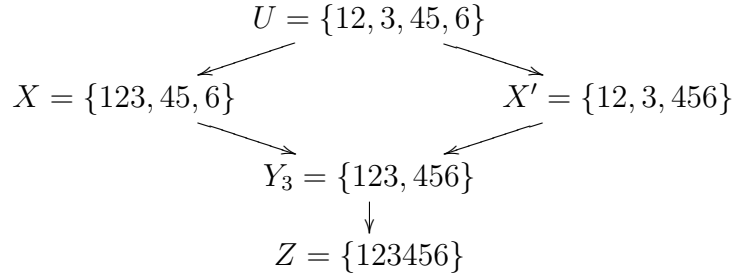
$$Y_3 = \{123, 456\}$$

$$Z = \{123456\}$$

in the quiver $\mathcal{Q}$. Then we have the relation

$$(U \to X \to Y_3 \to Z) + (U \to X' \to Y_3 \to Z) \in I.$$

Let $w(123456) = (456123)$. Then w maps U, X, Y_3 and Z to U, X', Y_3 and Z, respectively. Since $w(U) = \{45, 6, 12, 3\}$, it follows from Lemma 2.20 that $\sigma_U(w) = 1$. And since $\sigma_Z(w) = 1$, we have that

$$\sum_{v \in (S_6)^{(U \to X \to Y_3 \to Z)}} v(U \to X \to Y_3 \to Z)$$

$$= \sum_{v \in (S_6)^{(U \to X \to Y_3 \to Z)}} (vw)(U \to X' \to Y_3 \to Z)$$

$$= \sum_{v \in (S_6)^{(U \to X' \to Y_3 \to Z)}} v(U \to X' \to Y_3 \to Z).$$

Here we used the fact that $(S_6)^{(U \to X \to Y_3 \to Z)}w$ is a set of coset representatives for $(S_6)_{(U \to X' \to Y_3 \to Z)}$. Since the stabilizer of the path $(U \to X \to Y_3 \to Z)$ is equal to the stabilizer of that path $(U \to X' \to Y_3 \to Z)$,

$$\sum v(U \to X \to Y_3 \to Z)$$
$$= \frac{1}{2}\left(\sum v(U \to X \to Y_3 \to Z) + \sum v(U \to X' \to Y_3 \to Z)\right)$$
$$= \frac{1}{2}\sum v\Big((U \to X \to Y_3 \to Z) + (U \to X' \to Y_3 \to Z)\Big) \in I.$$

2.13 Future Directions

As mentioned at the outset, this chapter represents an installment of an ongoing project to study the descent algebra of W as a subalgebra of the face semigroup algebra of the reflection arrangement W. The two main outstanding tasks are determining the quiver of the descent algebra for arbitrary finite reflection groups, and then determining the quiver relations.

The results pesented here provide a great starting point. The quiver of $\mathcal{D}(W)$ has the quiver obtained from $\mathcal{Q}^W$ by reversing all the arrows as a subquiver, and we know there are no other vertices in the quiver. Therefore, it remains only to determine whether there are any other arrows. This is equivalent to determining when Lemma 2.15 holds for $p = 2$ for an arbitrary W.

As for the quiver relations, almost nothing is known about these even for $W = S_n$, so any information here would be new.

2.14 Appendix: Primitive Idempotents Revisited

In Section 1.5 a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ was recursively constructed from the poset $\mathcal{L}$. They were later used to construct a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$. In this section it will be shown that these idempotents can be constructed directly from the poset $\mathcal{L}/W$ using an analagous recursive construction.

The two settings are very similar. In the first, the support map $\mathrm{supp} : \mathcal{F} \to \mathcal{L}$ maps a basis of the algebra $k\mathcal{F}$ into the poset $\mathcal{L}$. The properties of this surjection were used to construct a complete system of primitive orthogonal idempotents in $k\mathcal{F}$ using lifts of $X \in \mathcal{L}$ under the support map. In the second setting, the elements $\sum_{y \in [x]} y$ form a linear basis for the algebra $(k\mathcal{F})^W$, and there is a natural way to map these elements into the poset $\mathcal{L}/W$. Namely, $\sum_{y \in [x]} y$ maps to the W-orbit of $\mathrm{supp}(x)$. The following Proposition shows that the same recursive construction applied to this map yields a complete system of primitive orthogonal idempotents in $(k\mathcal{F})^W$.

It would be interesting to determine conditions on a map from a basis of an algebra into a poset that would produce a complete system of primitive orthogonal idempotents using the above construction.

Proposition 2.21. *Suppose* $X \in \mathcal{L}$ *and* $x \in \mathcal{F}$ *with* $\mathrm{supp}(x) = X$. *Then the element* $\widehat{X} = \frac{1}{|\{y \in [x] : \mathrm{supp}(y) = X\}|} \sum_{y \in [x]} y$ *depends only on the* W-*orbit of* X. *For each* $[X] \in \mathcal{L}/W$ *let* $f_{[X]}$ *be defined by the recursion,*

$$f_{[X]} = \widehat{X} - \sum_{[Y] > [X]} \widehat{X} f_{[Y]}.$$

Then $\{f_{[X]}\}_{[X] \in \mathcal{L}/W}$ *is a complete system of primitive orthogonal idempotents for* $(k\mathcal{F})^W$. *In particular,* $f_{[X]} = \sum_{Y \in [X]} e_Y$, *where* e_Y *are the idempotents constructed*

in Example 2.6.

Proof. This follows by induction on $[X]$ in the poset $\mathcal{L}/W$. If $[X] = [\hat{1}]$, then $X = \hat{1}$ and $f_{[\hat{1}]} = \widehat{\hat{1}} = \frac{1}{|W|} \sum_{c \in \mathcal{C}} c = \hat{1}^+ = e_{\hat{1}}$. Let $[X] \in \mathcal{L}/W$ and suppose the result holds for all $[Y] > [X]$. Note that $\widehat{X} = \sum_{Z \in [X]} Z^+$, where Z^+ was defined in the previous example. Therefore,

$$f_{[X]} = \widehat{X} - \sum_{[Y]>[X]} \widehat{X} f_{[Y]} = \sum_{Z \in [X]} \left(Z^+ - \sum_{[Y]>[X]} Z^+ f_{[Y]} \right)$$

$$= \sum_{Z \in [X]} \left(Z^+ - \sum_{[Y]>[X]} Z^+ \sum_{U \in [Y]} e_U \right) = \sum_{Z \in [X]} \left(Z^+ - \sum_{[Y]>[X]} \sum_{\substack{U \in [Y] \\ U > Z}} Z^+ e_U \right)$$

$$= \sum_{Z \in [X]} \left(Z^+ - \sum_{U > Z} Z^+ e_U \right) = \sum_{Z \in [X]} e_Z. \qquad \square$$

THE SEMIGROUP ALGEBRA OF A LEFT REGULAR BAND

3.1 Introduction

The face semigroup algebra of a hyperplane arrangement is an example of a class of semigroups called left regular bands. These are semigroups S satisfying $x^2 = x$ and $xyx = xy$ for all $x, y \in S$. Much of the theory developed for the face semigroup algebra of a hyperplane arrangement extends to left regular bands and this chapter investigates these generalizations. In particular, we give a description of the quiver of semigroup algebra of a left regular band with identity in terms of equivalence classes of elements of the left regular band (Section 3.6). We also construct a complete system of primitive orthogonal idempotents in the semigroup algebra (Section 3.4), identify the projective indecomposable modules (Section 3.5) and give a description of the Cartan invariants (Section 3.11). To illustrate the theory we maintain two running examples throughout: the free left regular band; and the face semigroup algebra of a hyperplane arrangement. No assumptions are made on the characteristic of the field k.

The structure of this chapter is similar to the structure of Chapter 1.

3.2 Left Regular Bands

See [Brown, 2000, Appendix B] for foundations of left regular bands and for proofs of the statements presented in this section.

A *left regular band* is a semigroup S satisfying the following two properties.

(LRB1) $x^2 = x$ for all $x \in S$.

(LRB2) $xyx = xy$ for all $x, y \in S$.

Define a relation on the elements of S by $y \leq x$ iff $yx = x$. This relation is a partial order (reflexive, transitive and antisymmetric), so S is a poset.

Define another relation on the elements of S by $y \preceq x$ iff $xy = x$. This relation is reflexive and transitive, but not necessarily antisymmetric. Therefore we get a poset L by identifying x and y if $x \preceq y$ and $y \preceq x$. Let supp $: S \to L$ denote the quotient map. L is called the *support semilattice* of S and supp $: S \to L$ is called the *support map*.

Proposition 3.1. *If S is a left regular band, then there is a semilattice L and a surjection* supp $: S \to L$ *satisfying the following properties for all $x, y \in S$.*

1. *If $y \leq x$, then $\mathrm{supp}(y) \leq \mathrm{supp}(x)$.*

2. $\mathrm{supp}(xy) = \mathrm{supp}(x) \vee \mathrm{supp}(y)$.

3. $xy = x$ *iff* $\mathrm{supp}(y) \leq \mathrm{supp}(x)$.

4. *If S' is a subsemigroup of S, then the image of S' in L is the support semilattice of S'.*

Statement (1) says that supp is an order-preserving poset map. (2) says that supp is a semigroup map where we view L as a semigroup with product $\vee$. (3) follows from the construction of L, and (4) follows from the fact that (3) characterizes L upto isomorphism. If S has an identity element then L has a minimal element $\hat{0}$. If, in addition, L is finite, then L has a maximal element $\hat{1}$, and is therefore a lattice [Stanley, 1997, Proposition 3.3.1]. In this case L is the *support lattice* of S.

Example 3.2 (The Free Left Regular Band). The *free left regular band $F(A)$ with identity* on a finite set A is the set of all (ordered) sequences of distinct
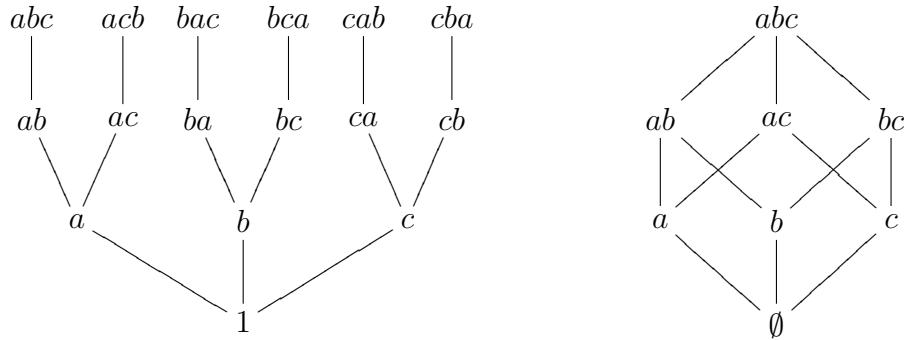
FIGURE 3.1: *The poset of the free left regular band $F(\{a, b, c\})$ on three generators and its support lattice.*

elements from A with multiplication defined by

$$(a_1, \ldots, a_l) \cdot (b_1, \ldots, b_m) = (a_1, \ldots, a_l, b_1, \ldots, b_m)^{\bowtie}$$

where $\bowtie$ means "delete any element that has occured earlier". Equivalently, $F(A)$ is the set of all words on the alphabet A that do not contain any repeated letters.

The empty sequence is an element of $F(A)$, therefore $F(A)$ contains an identity element. The support lattice of $F(A)$ is the lattice L of subsets of A and the support map supp : $F(A) \rightarrow A$ sends a sequence $(a_1, \ldots, a_l)$ to the set of elements in the sequence $\{a_1, \ldots, a_l\}$. Figure 3.1 shows the Hasse diagrams of the poset $(F(A), \leq)$ and the support lattice of $F(A)$, where $A = \{a, b, c\}$.

Example 3.3 (Hyperplane Arrangements). The face semigroup $\mathcal{F}$ of a hyperplane arrangement $\mathcal{A}$ is a left regular band. The support semilattice is the intersection semilattice $\mathcal{L}$ of the arrangement. The support map supp : $\mathcal{F} \rightarrow \mathcal{L}$ sends a face of the arrangement to the intersection of all the hyperplanes of the arrangement that contain the face. If $\mathcal{A}$ is a central arrangement, then the left regular band $\mathcal{F}$ contains an identity element.

3.3 Representations of the Semigroup Algebra

Let k denote a field and S a left regular band. The semigroup algebra of S is denoted by kS and consists of all formal linear combinations $\sum_{s \in S} \lambda_s s$, with $\lambda_s \in k$ and multiplication induced by $\lambda_s s \cdot \lambda_t t = \lambda_s \lambda_t st$, where st is the product of s and t in the semigroup S. The following summarizes Section 7.2 of [Brown, 2000].

Since S and L are semigroups and $\mathrm{supp} : S \to L$ is a semigroup morphism, the support map extends linearly to a surjection of semigroup algebras $\mathrm{supp} : kS \to kL$. The kernel of this map is nilpotent and the semigroup algebra kL is isomorphic to a product of copies of the field k, one copy for each element of L. Standard ring theory implies that $\ker(\mathrm{supp})$ is the Jacobson radical of kS and that the irreducible representations of kS are given by the components of the composition $kS \xrightarrow{\mathrm{supp}} kL \xrightarrow{\cong} \prod_{X \in L} k$. This last map sends $X \in L$ to the vector with 1 in the Y-position if $Y \geq X$ and 0 otherwise. The X-component of this surjection is the map $\chi_X : kS \to k$ defined on the faces $y \in S$ by

$$
\chi_X(y) = \begin{cases} 1, & \text{if } \mathrm{supp}(y) \leq X, \\ 0, & \text{otherwise.} \end{cases}
$$

The elements

$$
E_X = \sum_{Y \geq X} \mu(X, Y) Y \in kL, \tag{3.3.1}
$$

one for each $X \in L$, correspond to the standard basis vectors of $\prod_{X \in L} k$ under the isomorphism $kL \cong \prod_{X \in L} k$. They form a basis of kL and are a complete system of primitive orthogonal idempotents in kL.

3.4 Primitive Idempotents of the Semigroup Algebra

See Section 1.5 for the definition of a complete system of primitive orthogonal idempotents.

Let S denote a left regular band with identity. For each $X \in L$, fix an $x \in S$ with $\operatorname{supp}(x) = X$ and define elements in kS recursively by the formula,

$$e_X = x - \sum_{Y > X} x e_Y. \tag{3.4.1}$$

Lemma 3.4. *Let* $w \in S$ *and* $X \in L$. *If* $\operatorname{supp}(w) \not\leq X$, *then* $w e_X = 0$.

Proof. The proof of Lemma 1.4 only uses the left regular band structure of $\mathcal{F}$. $\square$

Theorem 3.5. *Let* S *denote a finite left regular band with identity and* L *its support lattice. Let* k *denote an arbitrary field. The elements* $\{e_X\}_{X \in L}$ *form a complete system of primitive orthogonal idempotents in the semigroup algebra* kS.

Proof. The proof of Theorem 1.5 only uses the left regular band structure of $\mathcal{F}$. $\square$

Remark 3.6. We can replace $x \in S$ in (3.4.1) with any linear combination $\tilde{x} = \sum_{\operatorname{supp}(x)=X} \lambda_x x$ of elements of support X whose coefficients λ_x sum to 1. The proofs still hold since the element $\tilde{x}$ is idempotent and satisfies $\operatorname{supp}(\tilde{x}) = X$ and $\tilde{x}y = \tilde{x}$ if $\operatorname{supp}(y) \leq X$. Unless explicitly stated we will use the idempotents constructed above.

Corollary 3.7. *The set* $\{x e_{\operatorname{supp}(x)} \mid x \in S\}$ *is a basis of* kS *of primitive idempotents.*

Proof. See Proposition 1.7. $\square$

3.5 Projective Indecomposable Modules of the Semigroup Algebra

For $X \in L$, let $S_X \subset S$ denote the set of elements of support X. For $y \in S$ and $x \in S_X$ define

$$
y \cdot x = \begin{cases} yx, & \operatorname{supp}(y) \leq \operatorname{supp}(x), \\ \\ 0, & \operatorname{supp}(y) \not\leq \operatorname{supp}(x). \end{cases}
$$

Then $\cdot$ defines an action of kS on the k-vector space kS_X spanned by S_X.

Lemma 3.8. *Let $X \in L$. Then $\{xe_X \mid \operatorname{supp}(x) = X\}$ is a basis for $(kS)e_X$.*

Proof. See Lemma 1.8. $\qquad\square$

Proposition 3.9. *There is a kS-module isomorphism $kS_X \cong kSe_X$ given by right multiplication by e_X. Therefore, the kS-modules kS_X are all the projective indecomposable kS-modules. The radical of kS_X is $\operatorname{span}_k\{y - y' \mid y, y' \in S_X\}$.*

Proof. See Proposition 1.9. $\qquad\square$

3.6 The Quiver of the Semigroup Algebra

Let A be a finite dimensional k-algebra whose simple modules are all one dimensional. The Ext-*quiver* or *quiver* of A is the directed graph with one vertex for each isomorphism class of simple modules and $\dim_k(\operatorname{Ext}^1_A(M_X, M_Y))$ arrows from X to Y, where M_X and M_Y are simple modules of the isomorphism classes corresponding to the vertices X and Y, respectively.

Let S be a left regular band with identity and let L denote the support lattice of S. Let $X, Y \in L$ with $Y \leq X$ and fix $y \in S$ with $\operatorname{supp}(y) = Y$. Define a relation

on the elements of S_X by $x \smile x'$ if there exists an element $w \in S$ satisfying $y < w$, $w < yx$ and $w < yx'$. (Equivalently, $yw = w$, $wx = yx$, $wx' = yx'$ and $\text{supp}(w) < X$.) Note that $x \smile x'$ iff $x \smile yx'$. Also note that for $X = \hat{1}$ and $Y = \hat{0}$, the relation becomes $x \smile x'$ iff there exists $w \neq 1$ such that $x > w$ and $x' > w$.

The relation $\smile$ is symmetric and reflexive, but not necessarily transitive. Let $\sim$ denote the transitive closure of $\smile$. Let $a_{XY} = {}^{\#}(S_X/\sim) - 1$, the number of equivalence classes of $\sim$ minus one. If $Y \not\leq X$, define $a_{XY} = 0$. In order to avoid confusion, we denote by a_{XY}^S the number a_{XY} computed in S. Since $u < v$ implies $yu < yv$ for all $u, v, y \in S$ (follows from (LRB2)), it follows that the relations $\smile$ and $\sim$ do not depend on the choice of y with $\text{supp}(y) = Y$.

Lemma 3.10. *Let S be a finite left regular band with identity and L its support lattice. Let M_X and M_Y denote the simple modules with irreducible characters χ_X and χ_Y, respectively. Then $\dim(\text{Ext}_A^1(M_X, M_Y)) = a_{XY}$.*

Proof. This proof is rather lengthy, so we banish it to a later section (3.15). $\square$

Theorem 3.11. *Let S be a left regular band with identity and L the support lattice of S. Let k denote a field. The quiver of the semigroup algebra kS has L as the vertex set and a_{XY} arrows from the vertex X to the vertex Y.*

3.7 An Inductive Construction of the Quiver

In this section we describe how knowledge about the numbers $a_{\hat{1}\hat{0}}^{S'}$ for certain sub-semigroups S' of S determine all the numbers a_{XY}^S. This allows for an inductive construction of the quiver of a left regular band.

Suppose S is a left regular band with identity. Let $X, Y \in L$ with $Y \leq X$ and let $y \in S$ be an element with $\text{supp}(y) = Y$. Then $yS = \{yw : w \in S\}$ and

$S_{\leq X} = \{w \in S : \mathrm{supp}(w) \leq X\}$ are subsemigroups of S.

Proposition 3.12. *Let S be a left regular band with identity, and let L denote the support lattice of S. Suppose $y \in S$ and $X \in L$. The quiver of the semigroup algebra $k(yS_{\leq X})$ of the left regular band $yS_{\leq X}$ is the full subquiver of the quiver of the semigroup algebra kS on the vertices in the interval $[\mathrm{supp}(y), X] \subset L$.*

The Proposition follows from the following Lemma that shows the number of arrows from X to Y in the quiver of kS is the number of arrows from $\hat{1}$ to $\hat{0}$ in the quiver of $k(yS_{\leq X})$, where $y \in S$ is any element of support Y. Recall that $a_{\hat{1}\hat{0}}^{yS_{\leq X}}$ denotes the number $a_{\hat{1}\hat{0}}$ computed in the left regular band $yS_{\leq X}$.

Lemma 3.13. *Let S be a left regular band with identity. Then $a_{XY}^S = a_{\hat{1}\hat{0}}^{yS_{\leq X}}$. That is, the number a_{XY} computed in S is the number $a_{\hat{1}\hat{0}}$ computed in $yS_{\leq X}$.*

Proof. If $\mathrm{supp}(y) \not\leq X$, then $yS_{\leq X}$ is empty. So $a_{X,Y}^S = 0 = a_{\hat{1}\hat{0}}^{yS_{\leq X}}$. So suppose $\mathrm{supp}(y) \leq X$.

Since $x \sim x'$ iff $x \sim yx'$ for any elements x, x' of support X, every equivalence class of $\sim$ (on S_X) contains an element of yS_X. Therefore, $a_{XY} + 1$ is the number of equivalence classes of $\sim$ restricted to yS_X.

Since $yS_{\leq X}$ is a subsemigroup of S, the support lattice of $yS_{\leq X}$ is the image of $yS_{\leq X}$ in L. Therefore, the support lattice of $yS_{\leq X}$ is the interval $[Y, X]$ in L. Since the top and bottom elements of $[Y, X]$ are X and Y respectively, the number $a_{\hat{1}\hat{0}}^{yS_{\leq X}} + 1$ is the number of equivalence classes of $\sim$ restricted to yS_X. $\qquad\square$

Therefore, if the numbers $a_{\hat{1}\hat{0}}^{yS_{\leq X}}$ are known for all the subsemigroups of S of the form $yS_{\leq X}$, then the quiver of kS is known. We illustrate this technique with two examples in the next two sections.

3.8 Example: The Free Left Regular Band

Let $S = F(A)$ denote the free left regular band on a finite set A (defined in Example 3.2). Recall that the support lattice L of S is the set of subsets of A.

Let $y \in S$ and let $Y \subset A$ denote the set of elements occuring in the sequence y. Then yS is the set of all sequences of elements of A (without repetition) that begin with the sequence y. Therefore, yS is isomorphic to the free left regular band on $A \backslash Y$. If $X \subset A$ (so $X \in L$), then $S_{\leq X}$ is the set of all sequences containing only elements from X (without repetition). Therefore, $S_{\leq X}$ is also a free left regular band. It follows that $yS_{\leq X}$ is a free left regular band for any $y \in S$ and $X \subset A$. Therefore, the quiver of S is determined once the numbers $a_{\hat{0}\hat{1}} = a_{A\emptyset}$ are known for any free left regular band.

If two sequences $x, y \in S$ begin with the same element $a \in A$, then $ax = x$ and $ay = y$. Therefore, $x \sim y$. Conversely, if $x \smile y$, then there is a nonempty sequence w such that $wx = x$ and $wy = y$. Then x and y both begin with the first element of w. Therefore, $x \sim y$ iff x and y are sequences begining with the same element. So the equivalence classes of $\sim$ are determined by the first elements of the sequences in S. Hence, $a_{\hat{1}\hat{0}} = {}^{\#}(A) - 1$. This argument applies to any free left regular band with identity, so $a_{XY} = {}^{\#}(X \backslash Y) - 1$ since $yS_{\leq Y}$ is isomorphic to the free left regular band on the elements $X \backslash Y$.

Theorem 3.14. (K. S. Brown, private communication.) *Let $S = F(A)$ be the free left regular band on a finite set A and let k denote a field. Then the quiver of the semigroup algebra kS has one vertex X for each subset X of A and ${}^{\#}(X \backslash Y) - 1$ arrows from X to Y if $Y \subset X$ (and no other arrows or vertices).*
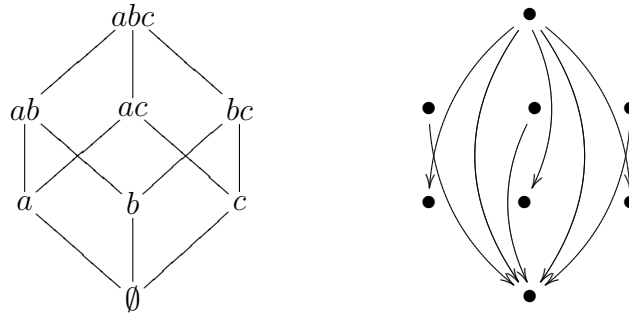
FIGURE 3.2: *The support lattice and the quiver of the semigroup algebra of the free left regular band on three generators. See Figure 3.1.*

3.9 Example: The Face Semigroup of a Hyperplane Arrangement

Let $\mathcal{F}$ be the face semigroup of a central hyperplane arrangement $\mathcal{A}$ and let $\mathcal{L}$ be the intersection lattice of $\mathcal{A}$ (see Example 3.3.) Let $X, Y \in \mathcal{L}$ and y a face of support Y. Then the subsemigroup $y\mathcal{F}_{\leq X}$ is the semigroup of faces of a hyperplane arrangement with intersection lattice $[Y, X] \subset \mathcal{L}$. (Specifically, this hyperplane arrangement is given by $\{X \cap H : H \in \mathcal{A}, Y \subset H, X \not\subset H\}$.) Therefore, we know all the numbers a_{XY} for $\mathcal{F}$ if we know the number $a_{\hat{1}\hat{0}}$ for the face semigroup of an arbitrary arrangement.

If $\mathcal{L}$ contains only one element, then $\hat{0} = \hat{1}$ and $a_{\hat{1}\hat{0}} = 0$. Suppose that $\mathcal{L}$ contains at least two elements. It is well-known that for any two distinct chambers c and d, there exists a sequence of chambers $c_0 = c, c_1, \ldots, c_i = d$ such that c_{j-1} and c_j share a common codimension one face w_j for each $1 \leq j \leq i$ [Brown, 1989, §I.4E Proposition 3]. Therefore, $c_{j-1} \smile c_j$ unless w_j is of support $\hat{0}$, in which case $\mathcal{L}$ has

two elements. Equivalently, $c \sim d$ iff the arrangement is of rank greater than 2. So if $\mathcal{L}$ has exactly two elements, then $a_{\hat{1}\hat{0}} = 1$ and if $\mathcal{L}$ has more than two elements then $a_{\hat{1}\hat{0}} = 1$.

Theorem 3.15 (Theorem 1.19). *The quiver Q of the semigroup algebra $k\mathcal{F}$ coincides with the Hasse diagram of $\mathcal{L}$. That is, there is exactly one arrow $X \to Y$ iff $Y \lessdot X$.*

3.10   Idempotents in the subalgebras $k(yS)$ and $kS_{\geq X}$

This section describes the subalgebras of kS generated by the subsemigroups yS and $S_{\leq Y}$ of S.

Let S be a left regular band. Recall that for $y \in S$, the set $yS = \{yw : w \in S\} = \{w \in S : w > y\}$ is a subsemigroup of S. Note that if $\text{supp}(y') = \text{supp}(y)$ then the semigroups yS and $y'S$ are isomorphic with isomorphism given by multiplication by y (the inverse is multiplication by y'). Since yS is a subsemigroup of S, the support lattice of yS is the image of yS in L by Propositon 3.1, which is the interval $[Y, \hat{1}]$.

Proposition 3.16. *Let S be a left regular band, let $y \in S$ and let $Y = \text{supp}(y)$. There exists a complete system of primitive orthogonal idempotents $\{e_X : X \in L\}$ in kS such that $\{e_X : X \geq Y\}$ is a complete system of primitive orthogonal idempotents in the semigroup algebra $k(yS)$. Moreover, $k(yS) = (\sum_{X \geq Y} e_X)kS$.*

Proof. For each $X \in L$, fix $x \in S$ with $\text{supp}(x) = X$. If $X \geq Y$, then replace x with yx. Note that $\text{supp}(yx) = \text{supp}(x)$ since $X \geq Y$. Therefore, $x > y$ if $X \geq Y$. The formula $e_X = x - \sum_{W > X} xe_W$ for $X \in L$ defines a complete system of primitive orthogonal idempotents for kS (Theorem 3.5). And since the support lattice of yS

is $[Y, \hat{1}] \subset L$, the elements $e_X = x - \sum_{W > X} xe_W$ for $X \geq Y$ define a complete system of primitive orthogonal idempotents in $k(yS)$. Since y is the identity of yS, we have $y = \sum_{X \geq Y} e_X$. Therefore, $k(yS) = y(kS) = (\sum_{X \geq Y} e_X)kS$. $\qquad\square$

If $Y \in L$, then $S_{\leq Y} = \{w \in S : \mathrm{supp}(w) \leq Y\}$ is a subsemigroup of S. The support lattice of $S_{\leq Y}$ is the interval $[\hat{0}, Y]$ of L. Let $\mathrm{proj}_{kS_{\leq Y}} : kS \to kS_{\leq X}$ denote the projection onto the subspace $kS_{\leq X}$ of kS.

Proposition 3.17. *Let S be a left regular band and $Y \in L$. Let $\{e_X : X \in L\}$ denote a complete system of primitive orthogonal idempotents of kS. Then $\{\mathrm{proj}_{kS_{\leq Y}}(e_X) : X \leq Y\}$ is a complete system of primitive orthogonal idempotents of $kS_{\leq Y}$. Moreover, the semigroup algebra $k(S_{\leq Y})$ is isomorphic to $kS(\sum_{X \leq Y} e_X)$.*

Proof. The map $\mathrm{proj}_{kS_{\leq Y}}$ is an algebra morphism $kS \to kS_{\leq Y}$. This follows from the fact that $\mathrm{supp}(wx) = \mathrm{supp}(w) \vee \mathrm{supp}(x)$ for any $x, w \in S$. So if $X \leq Y$, then $\mathrm{proj}_{kS_{\leq Y}}(e_X) = x - \sum_{W > X} x\, \mathrm{proj}_{kS_{\leq Y}}(e_W)$ since $e_X = x - \sum_{W > X} xe_W$. Therefore, the elements $\mathrm{proj}_{kS_{\leq Y}}(e_X)$ for $X \leq Y$ form a complete system of primitive orthogonal idempotents for the semigroup algebra of the left regular band $S_{\leq Y}$ (Theorem 3.5). Since $\mathrm{proj}_{kS_{\leq Y}}$ is an algebra morphism, it restricts to a surjective morphism of algebras $\mathrm{proj}_{kS_{\leq Y}} : kS(\sum_{X \leq Y} e_X) \to k(S_{\leq Y})$. Since $kS_X \cong (kS)e_X$ for all $X \in L$ as kS-modules (Proposition 3.9), $\dim(kS_{\leq Y}) = \dim(\sum_{X \leq Y}(kS)e_X)$. So $\mathrm{proj}_{kS_{\leq Y}}$ is an isomorphism. Its inverse is right multiplication by $\sum_{X \leq Y} e_X$. $\qquad\square$

3.11 Cartan Invariants of the Semigroup Algebra

Recall that the *Cartan invariants* of a finite dimensional k-algebra A are the numbers $\dim_k(\mathrm{Hom}_A(Ae_X, Ae_Y))$, where $\{e_X\}_{X \in I}$ is a complete system of primitive orthogonal idempotents for A. They are independent of the choice of $\{e_X\}_{X \in I}$.

Let S be a left regular band with identity and let L denote the support lattice of S. For $X, Y \in L$, define numbers $m(Y, X)$ follows. If $Y \not\leq X$, then $m(Y, X) = 0$. If $Y \leq X$, then define $m(Y, X)$ by the formulas

$$\sum_{W \leq Y \leq X} m(Y, X) = {}^{\#}(wS_X),$$

one for each $W \in L$, where w is an element of support W. Recall that the number ${}^{\#}(wS_X)$ does not depend on the choice of w with $\mathrm{supp}(w) = W$. Equivalently,

$$m(Y, X) = \sum_{Y \leq W \leq X} \mu(Y, W) \; {}^{\#}(wS_X),$$

where μ is the Möbius function of L [Stanley, 1997, §3.7].

Proposition 3.18. *Let S be a left regular band with identity. Let $\{e_X\}_{X \in L}$ denote a complete system of primitive orthogonal idempotents for kS. Then for any X, Y,*

$$\dim(e_Y kSe_X) = \dim \mathrm{Hom}_{kS}(kSe_Y, kSe_X) = m(Y, X).$$

Therefore, the numbers $m(Y, X)$ are the Cartan invariants of kS.

Proof. The first equality follows from the identity $\mathrm{Hom}_A(Ae, Af) \cong eAf$ for idempotents e, f of a k-algebra A. If $Y \not\leq X$, then it follows from (LRB2) and Lemma 3.4 that $e_Y kSe_X = 0$. Suppose that $Y \leq X$. From the previous section, $k(yS) = \sum_{W \geq Y} e_W kS$ for some complete system of primitive orthogonal idempotents. Combined with the isomorphism $kS_X \cong kSe_X$ we get $k(yS_X) \cong \bigoplus_{Y \leq W \leq X} e_W kSe_X$. Therefore,

$$\sum_{Y \leq W \leq X} m(W, X) = \dim(k(yS_X)) = \sum_{Y \leq W \leq X} \dim(e_W kSe_X).$$

The result now follows by induction. If $X = Y$, then $\dim e_X kSe_X = m(X, X)$. Suppose the result holds for all W with $Y < W \leq X$. Then

$$\dim e_Y kSe_X = \sum_{Y \leq W \leq X} m(W, X) - \sum_{Y < W \leq X} \dim e_W kSe_X$$

$$= \sum_{Y \leq W \leq X} m(W, X) - \sum_{Y < W \leq X} m(W, X)$$

$$= m(Y, X). \qquad \qquad \square$$

3.12 The Cartan Invariants for Hyperplane Arrangements

Let $\mathcal{F}$ denote the semigroup of faces of a hyperplane arrangement $\mathcal{A}$. Then $^{\#}(w\mathcal{F}_X)$ is the number of faces of support X containing w as a face. Zaslavsky's Theorem [Zaslavsky, 1975] gives that this is $\sum_{W \leq Y \leq X} |\mu(Y, X)|$. Therefore, the Cartan invariants of $k\mathcal{F}$ are $m(Y, X) = |\mu(Y, X)|$. This was also proved in Proposition 1.11.

3.13 The Cartan Invariants for a Free Left Regular Band

Let S be a free left regular band on the finite set A. The support lattice of S is the lattice of subsets of A. Therefore, $\mu(Y, W) = (-1)^{\#(W \backslash Y)}$ [Stanley, 1997, Example 3.8.3] for any $Y, W \in L$. And $^{\#}(wS_X) = {}^{\#}(X \backslash W)!$ since the number of elements of maximal support in the free left regular band on A is precisely $^{\#}A!$. If $n = {}^{\#}X$ and $j = {}^{\#}Y$, and $Y \subset X$, then

$$m(Y, X) = \sum_{Y \leq W \leq X} \mu(Y, W) \, {}^{\#}(wS_X)$$

$$= \sum_{Y \leq W \leq X} (-1)^{\#W - j} \, (n - {}^{\#}W)!$$

$$= \sum_{i=j}^{n} \sum_{\substack{Y \subseteq W \subseteq X \\ \#W = i}} (-1)^{i-j}(n-i)!$$

$$= \sum_{i=j}^{n} (-1)^{i-j}(n-i)! \binom{n-j}{i-j}$$

$$= (n-j)! \sum_{i=j}^{n} \frac{(-1)^{i-j}}{(i-j)!}$$

$$= (n-j)! \sum_{i=0}^{n-j} \frac{(-1)^i}{i!}.$$

Therefore, the number $m(Y,X)$ depends only on the cardinality of $X\backslash Y$ and we denote it by m_i where $i = {}^{\#}(X\backslash Y)$.

We will now prove that these numbers count paths in the quiver of kS. For a set A of cardinality n, let Q_n be the directed graph with one vertex for each subset of A and ${}^{\#}(X\backslash Y) - 1$ arrows from X to Y if $Y \subset X$. Let p_n denote the number of paths in Q_n beginning at A and ending at $\emptyset$. Note that if $Y \subset X \subset A$, then the number of paths beginning at X and ending at Y in Q_n is p_m where $m = {}^{\#}(X\backslash Y)$.

For each $0 \le i \le n-1$ there are $n-i-1$ arrows from A to sets of size i, and there are $\binom{n}{i}$ such sets, so $p_n = \sum_{0 \le i \le n-1} \binom{n}{i}(n-i-1)p_i$ for $n \ge 1$. Equivalently, by splitting $n-i-1$,

$$\sum_{0 \le i \le n} \binom{n}{i} p_i = \sum_{0 \le i \le n-1} \binom{n}{i}(n-i)p_i.$$

If the m_i satisfy the above recurrence, then $m_i = p_i$ for all i since $m_0 = 1 = p_0$.

$$\sum_{0 \le i \le n-1} \binom{n}{i}(n-i)m_i = \sum_{0 \le i \le n-1} \frac{n!}{(n-i-1)!\, i!} m_i$$

$$= \sum_{0 \le i \le n-1} \frac{n!}{(n-i-1)!\, i!} \left(i! \sum_{0 \le j \le i} \frac{(-1)^j}{j!} \right)$$

$$= \sum_{0 \le i \le n-1} \frac{n!}{(n-i-1)!} \left(\sum_{0 \le j \le i} \frac{(-1)^j}{j!} \right)$$

$$= \sum_{1 \le k \le n} \frac{n!}{(n-k)!} \left(\sum_{0 \le j \le k-1} \frac{(-1)^j}{j!} \right)$$

$$= \sum_{1 \le k \le n} \frac{n!}{(n-k)!} \left(\sum_{0 \le j \le k} \frac{(-1)^j}{j!} - \frac{(-1)^k}{k!} \right)$$

$$= \sum_{1 \le k \le n} \frac{n!}{(n-k)!} \left(\sum_{0 \le j \le k} \frac{(-1)^j}{j!} \right) - \sum_{1 \le k \le n} \frac{n!}{(n-k)!} \left(\frac{(-1)^k}{k!} \right)$$

$$= \sum_{1 \le k \le n} \binom{n}{k} \left(k! \sum_{0 \le j \le k} \frac{(-1)^j}{j!} \right) - \sum_{1 \le k \le n} \binom{n}{k}(-1)^k$$

$$= \sum_{1 \leq k \leq n} \binom{n}{k} m_k + 1$$

$$= \sum_{1 \leq k \leq n} \binom{n}{k} m_k + \binom{n}{0} m_0$$

$$= \sum_{0 \leq k \leq n} \binom{n}{k} m_k.$$

Theorem 3.19. (K. S. Brown, private communication.) *Let $S = F(A)$ be the free left regular band on a finite set A. Then $kS \cong kQ$, where kQ is the path algebra of the quiver Q of kS.*

Proof. Since Q is the quiver of kS, there is an algebra surjection $kQ \to kS$, where kQ is the path algebra of Q. The canonical basis for kQ is the set of paths in Q, so using the fact that $m(Y, X) = \dim(e_Y kS e_X)$ counts the number of paths in Q from X to Y, we have $\dim(kQ) = \sum_{Y,X} m(Y, X) = \sum_{Y,X} \dim(e_Y kS e_X) = \dim(kS)$. $\square$

3.14 Future Directions

Problem 1. Although this chapter successfully determines the quiver of the semigroup algebra of a left regular band, it says nothing about the quiver relations. Describe the quiver relations of the semigroup algebra of a left regular band with identity.

Problem 2. We proved that the face semigroup algebra of a hyperplane arrangement is a Koszul algebra. Since this is the semigroup algebra of a left regular band, it is natural to ask this question for all left regular bands. Determine which left regular bands give Koszul semigroup algebras.

Problem 3. Another nice property of the face semigroup algebra of a hyperplane arrangement is that the quiver of the semigroup algebra coincides with the support lattice of the semigroup. In fact, the support lattice completely determines the

semigroup algebra. Determine the left regular bands S for which the quiver of kS coincides with the support lattice of L. (From our description of the quiver of kS, we have a description of these left regular bands in terms of the equivalence classes of $\sim$.) Determine those S for which the support lattice L completely determines kS.

Problem 4. A *band* is a semigroup B satisfying $b^2 = b$ for all $b \in B$. Left regular bands are examples of bands and it would be interesting to generalize the above results to arbitrary bands. Describe the quiver of the semigroup algebra kB of a band B with identity. Construct a complete system of primitive orthogonal idempotents for kB. Determine the bands B for which kB is a Koszul algebra.

3.15   Appendix: Proof of Lemma 3.10

Lemma 3.10. *Let S be a finite left regular band with identity and L its support lattice. Let M_X and M_Y denote the simple modules with irreducible characters χ_X and χ_Y, respectively. Then $\dim(\mathrm{Ext}_A^1(M_X, M_Y)) = a_{XY}$.*

Proof. As a vector space $M_X = k$ and the action of kS on M_X is given by χ_X: if $y \in S$ and $\lambda \in k$, then $y \cdot \lambda = \chi_X(y)\lambda$.

Since the following is a short exact sequence of kS-modules with kS_X projective,

$$0 \longrightarrow \ker(\chi_X|_{kS}) \longrightarrow kS_X \xrightarrow{\chi_X} M_X \longrightarrow 0$$

Proposition 7.2 in Chapter V of [Cartan and Eilenberg, 1999] gives the exact sequence

$$\mathrm{Hom}_{kS}(kS_X, M_Y) \longrightarrow \mathrm{Hom}_{kS}(\ker(\chi_X|_{kS}), M_Y) \longrightarrow \mathrm{Ext}_{kS}^1(M_X, M_Y) \longrightarrow 0.$$

Let K denote the kernel of $\chi_X|_{kS_X}$. Then K is spanned by the differences of elements of support X. If $f \in \mathrm{Hom}_{kS}(K,\ M_Y)$ and x, x' are elements of support X, then $f(x - x') = 1f(x - x') = \chi_Y(y)f(x - x') = y \cdot f(x - x') = f(y \cdot (x - x'))$, for any element y of support Y. So if $Y \nleq X$ or if $Y = X$, then $f = 0$. Therefore, $\mathrm{Hom}_{kS}(K, M_Y) = 0$ if $Y \nless X$. It follows that

$$\mathrm{Ext}_{kS}^1(M_X, M_Y) = 0 = a_{XY} \text{ for } Y \nless X.$$

Suppose $Y < X$. If $f \in \mathrm{Hom}_{kS}(kS_X, M_Y)$, then for all $x \in S_X$, $f(x) = f(x^2) = f(x \cdot x) = x \cdot f(x) = \chi_Y(x)f(x) = 0f(x) = 0$ for all $x \in S$ with $\mathrm{supp}(x) = X$. Therefore, $\mathrm{Hom}_{kS}(kS_X, M_Y) = 0$. Hence,

$$\mathrm{Ext}_{kS}^1(M_X, M_Y) \cong \mathrm{Hom}_{kS}(K, M_Y) \text{ for } Y < X.$$

Suppose $x \smile x'$. Then there exists a $w \in S$ with $y < w$, $\text{supp}(w) < X$, $wx = yx$ and $wx = yx'$. Then $x - x' \in K$, and for any $f \in \text{Hom}_{kS}(K, M_Y)$ we have $f(x - x') = \chi_Y(y)f(x - x') = f(yx - yx') = f(wx - wx') = f(w \cdot (x - x')) = w \cdot f(x - x') = \chi_Y(w)f(x - x') = 0f(x - x') = 0$. Therefore, $f(x - x') = 0$ if $x \smile x'$. If $x \sim x'$, then there exist $x_0 = x, x_1, \ldots, x_i = x'$ such that $x_{j-1} \smile x_j$ for $1 \le j \le i$, and $f(x - x') = f(x_0 - x_1) + f(x_1 - x_2) + \cdots + f(x_{i-1} + x_i) = 0$. Therefore, $f(x - x') = 0$ if $x \sim x'$. So f can only be nonzero on differences of elements in different equivalence classes of $\sim$. Moreover, the equivalence classes determine f: if $u \sim x$ and $u' \sim x'$, then $f(u - u') = f(u - x) + f(x - x') + f(x' - u') = f(x - x')$. Therefore,

$$\dim(\text{Ext}^1_{kS}(M_X, M_Y)) = \dim(\text{Hom}_{kS}(K, M_Y)) \le a_{XY}.$$

Fix y with $\text{supp}(y) = Y$ and let $x, x' \in S_X$ with $x \nsim x'$. Since $\{u - x : u \ne x, \text{supp}(u) = X\}$ is a basis for K, we get a well-defined linear function $f : K \to k$ by defining

$$f(u - x) = \begin{cases} 1, & \text{if } u \sim x', \\ 0, & \text{otherwise.} \end{cases}$$

We now show that $f : K \to M_Y$ is a kS-module map. That is, $f(w \cdot (u - x)) = \chi_Y(w) \cdot f(u - x)$ for all $w \in S$ and for all $u \in S_X$.

Suppose $\text{supp}(w) \nleq Y$. Then $w \cdot f(u - x) = 0$ since w acts trivially on M_Y. If $\text{supp}(w) \nless X$, then w acts trivially on K and so $w \cdot f(u - x) = 0 = f(w \cdot (u - x))$. So suppose $\text{supp}(w) < X$. Then $f(w \cdot (u - x)) = f(wu - wx) = f(wu - x) - f(wx - x)$. Since $v \sim x'$ iff $yv \sim x'$ for any $v \in S_X$, it follows that $f(wu - x) = f(ywu - x)$ and $f(wx - x) = f(ywx - x)$. If $\text{supp}(yw) = X$, then $ywu = yw = ywx$ (LRB2), so $f(w \cdot (u - x)) = 0$. If $\text{supp}(yw) < X$, then we have an element $v = yw$ satisfying

$v > y$, $\mathrm{supp}(v) < X$, $v(wu) = y(wu)$ and $v(wx) = y(wx)$. That is, $wu \sim wx$ and it follows that $f(wu - x) = f(wx - x)$. So $f(w \cdot (u - x)) = 0$.

Suppose $\mathrm{supp}(w) \leq Y$. Then w acts as the identity on M_Y. Hence, $w \cdot f(u-x) = f(u - x)$. Since $\mathrm{supp}(w) \leq Y$ and $Y \leq X$, we have that $\mathrm{supp}(w) \leq X$. Therefore, $f(w \cdot (u - x)) = f(wu - wx) = f(wu - x) - f(wx - x)$. Since $v \sim x'$ iff $yv \sim x'$, we have $f(wu - x) = f(y(wu) - x) = f(yu - x) = f(u - x)$ since $\mathrm{supp}(w) \leq Y$. Similarly, $f(wx - x) = f(x - x) = 0$. Therefore, $f(w \cdot (u - x)) = f(u - x)$.

This establishes that $f : K \to M_Y$ is a kS-module map. And since f is nonzero only on differences of the form $u - u'$ with $u \sim x$ and $u' \sim x'$, there are exactly a_{XY} such kS-module maps. These maps are linearly independent, therefore

$$\dim(\mathrm{Ext}^1_{kS}(M_X, M_Y)) = \dim(\mathrm{Hom}_{kS}(K, M_Y)) \geq a_{XY}. \;\square$$

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