The Face Semigroup Algebra of a Hyperplane Arrangement

A Thesis Defence

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Hyperplane Arrangements

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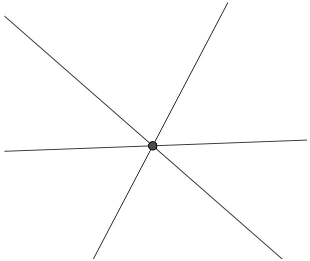

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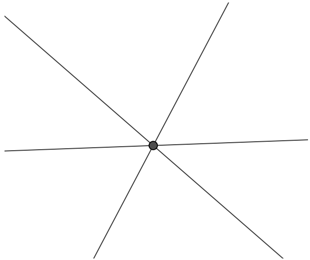

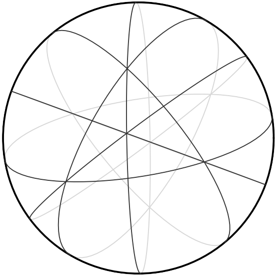

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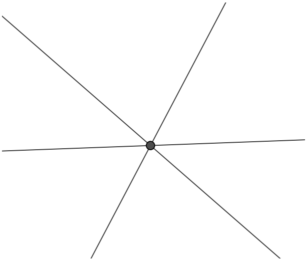

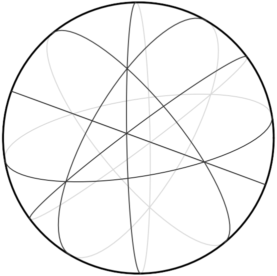

Assumption: all hyperplanes contain $0 \in \mathbb{R}^n$.

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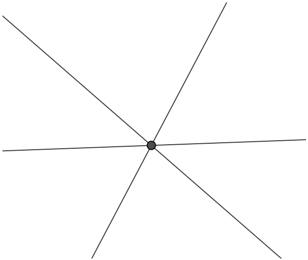

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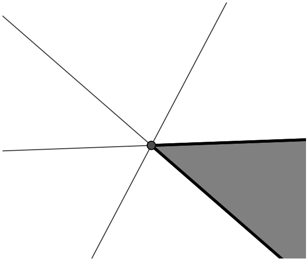

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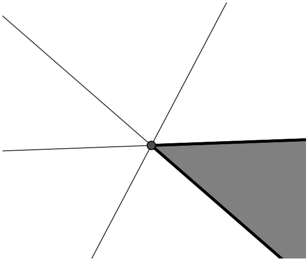

The set of faces of these polyhedra are the faces of $\mathcal{A}$.

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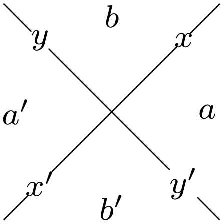

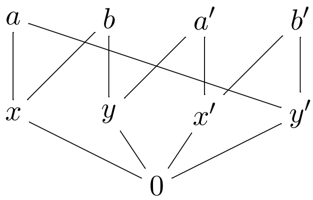

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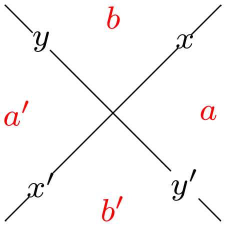

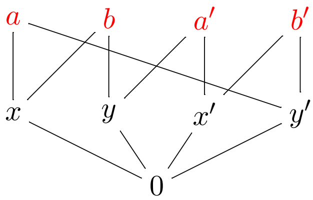

The maximal faces are called chambers.

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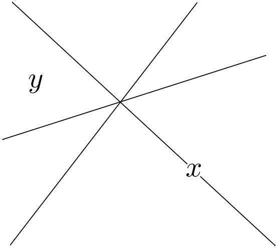

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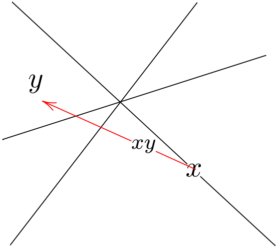

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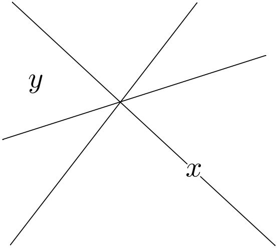

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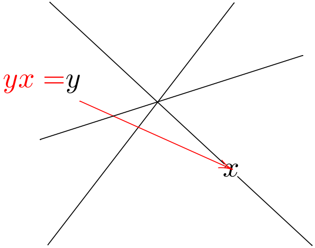

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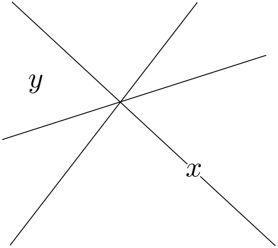

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- Get information about eigenvalues and multiplicities, diagonalization, ...

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- Associated to W is a hyperplane arrangement $\mathcal{A}(W)$ consisting of the hyperplanes fixed by some reflection in W.
- Bidigare (1997) showed that the descent algebra is a subalgebra of $k\mathcal{F}$ (for the arrangement $\mathcal{A}(W)$).

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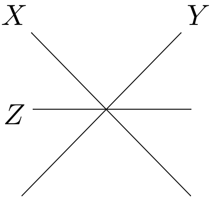

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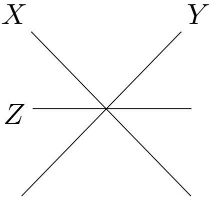

$\mathbb{R}^n$

Intersection of no hyperplanes.

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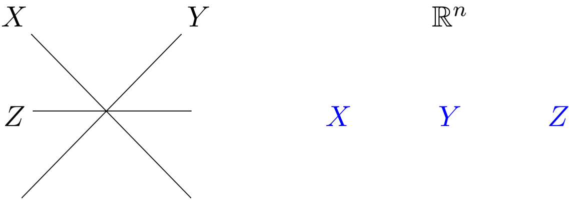

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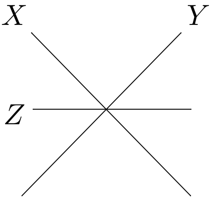

$$\mathbb{R}^n$$

$$X \qquad Y \qquad Z$$

$$0$$

Intersection of at least two hyperplanes.

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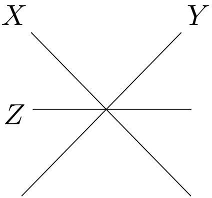

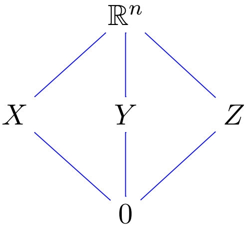

Order by inclusion.

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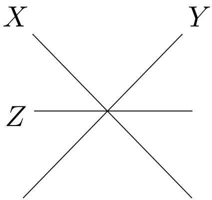

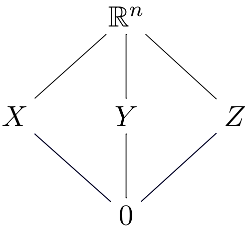

Warning: Some order $\mathcal{L}$ by *reverse* inclusion!

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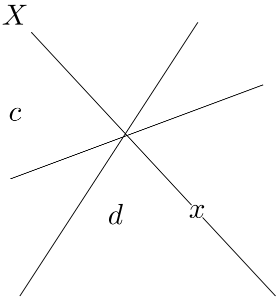

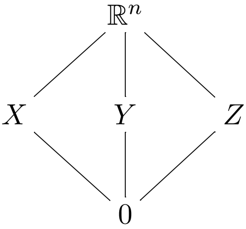

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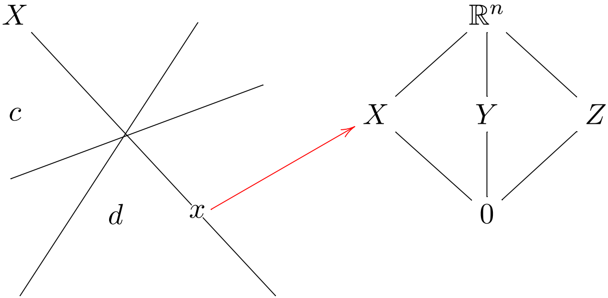

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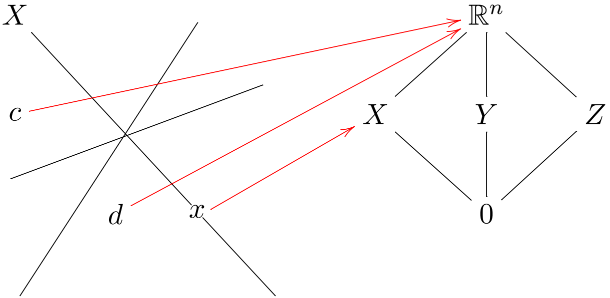

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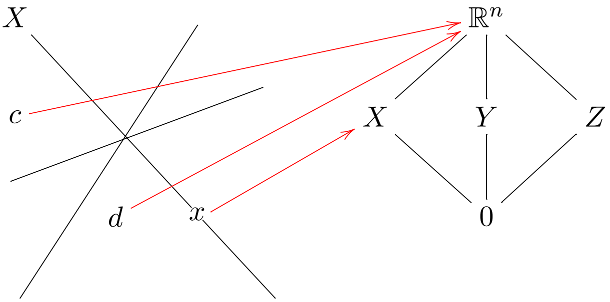

supp is an order-preserving surjection of posets.

Simple $k\mathcal{F}$-modules

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- Therefore, $k\mathcal{F}$ comes from a *quiver*.

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- The arrows $X \to Y$ correspond to a basis of

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- φ is surjective.
- So $(Q, \ker\varphi)$ is a presentation of A.

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- To compute $\operatorname{Ext}^p_{k\mathcal{F}}(S_X, S_Y)$ we need a projective resolution of S_X: an exact sequence of projective $k\mathcal{F}$-modules.

$$\cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S_X \longrightarrow 0$$

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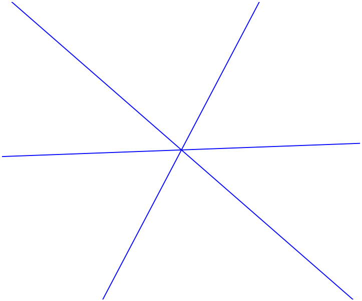

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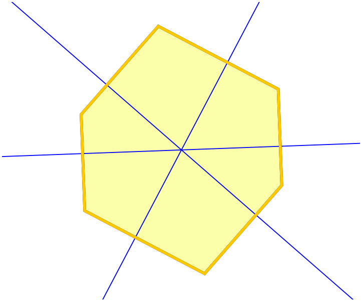

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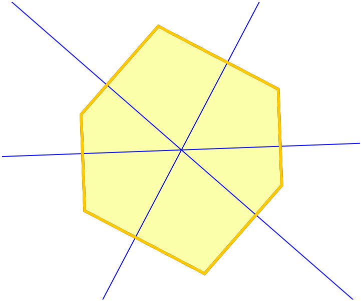

Associated to the arrangement is a *zonotope* Z.

The face poset of Z is the opposite poset of $\mathcal{F}$.

Augmented Cellular Chain Complex

$$\cdots \xrightarrow{\partial} k\mathcal{F}_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_0 \xrightarrow{\chi} k \longrightarrow 0,$$

This is the augmented cellular chain complex of Z, where $\mathcal{F}_p$ is the set of codimension p faces in $\mathcal{F}$.

Exactness

$$\cdots \xrightarrow{\partial} k\mathcal{F}_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_0 \xrightarrow{\chi} k \longrightarrow 0$$

The sequence is exact because the homology of Z is trivial.

$k\mathcal{F}$-module Structure

$$\cdots \xrightarrow{\partial} k\mathcal{F}_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_0 \xrightarrow{\chi} k \longrightarrow 0$$

- The vector spaces $k\mathcal{F}_p$ are $k\mathcal{F}$-modules via the action

$$x \cdot y = \begin{cases} xy, & \text{if } \operatorname{supp}(x) \leq \operatorname{supp}(y), \\ 0, & \text{if } \operatorname{supp}(x) \nleq \operatorname{supp}(y). \end{cases}$$

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- So $k\mathcal{F}_p$ is projective: $\quad k\mathcal{F}_p \cong \bigoplus_{\dim(X)=p} k\mathcal{F}e_X.$

Boundary Morphisms

$$\cdots \xrightarrow{\partial} k\mathcal{F}_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_0 \xrightarrow{\chi} k \longrightarrow 0$$

With this action the boundary operators are module morphisms.

The Augmentation Map

$$\cdots \xrightarrow{\quad \partial \quad} k\mathcal{F}_p \xrightarrow{\quad \partial \quad} \cdots \xrightarrow{\quad \partial \quad} k\mathcal{F}_0 \xrightarrow{\quad \chi \quad} k \xrightarrow{\qquad} 0$$

- The augmentation χ sends a chamber $c \in \mathcal{F}_0$ to 1.

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- So we get a projective resolution of $S_{\mathbb{R}^n}$.

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- Using properties of the idempotents e_X this projective resolution gives a projective resolution of the simple module S_X over $k\mathcal{F}$.

The Ext Groups

- The Ext-groups are

$$\dim\left(\mathrm{Ext}^p_{k\mathcal{F}}(S_X, S_Y)\right) = \begin{cases} 1, & \text{if } \mathrm{codim}_X(Y) = p, \\ 0, & \text{otherwise.} \end{cases}$$

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- $p = 1$: There is exactly one arrow $X \to Y$ iff $Y \lessdot X$.
- $p = 2$: There is one relation for each interval of length two; the sum of the paths of length two in that interval.

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- The Koszul dual of $k\mathcal{F}$ is the incidence algebra $I(\mathcal{L}^*)$.

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- W acts on the faces $\mathcal{F}$, hence also on $k\mathcal{F}$.
- Bidigare showed that the invariant subalgebra $(k\mathcal{F})^W$ is isomorphic to *Solomon's descent algebra*.

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2. If $W = S_n$, then $\mathcal{Q}^{S_n}$ is the quiver of $(k\mathcal{F})^{S_n}$.

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- The semigroup algebra kS also comes from a quiver Q.

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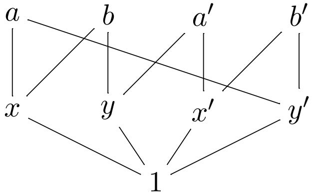

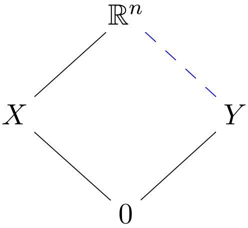

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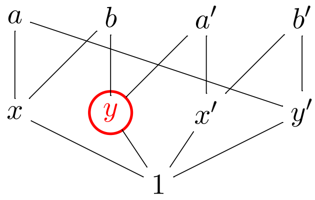

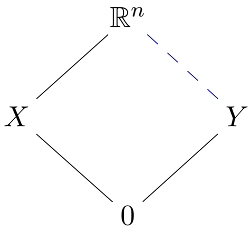

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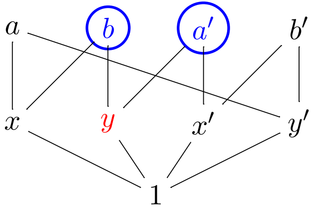

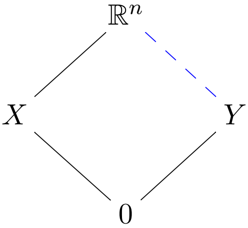

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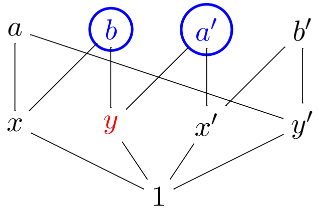

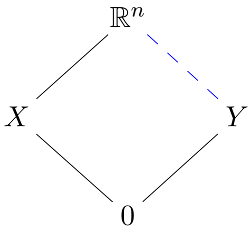

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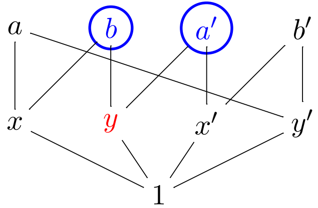

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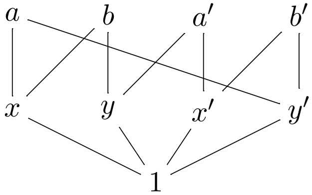

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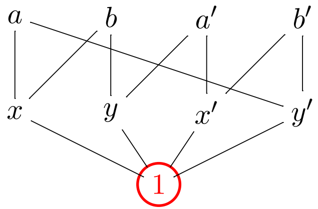

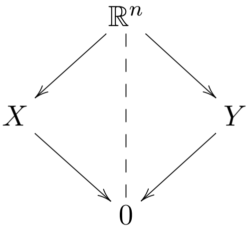

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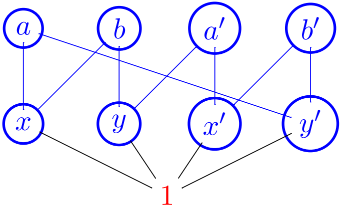

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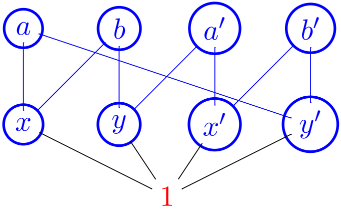

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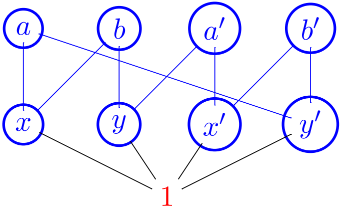

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- Study the *twisted group algebra* $k\mathcal{F} * kW$.
- Characterize the (left regular) bands that give Koszul algebras.
- *Interval Greedoids*: a generalization of matroid that includes antimatroids. Develop an "oriented interval greedoid" theory.