The Descent Algebra, Geometrically

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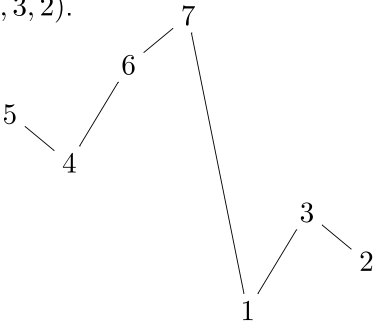

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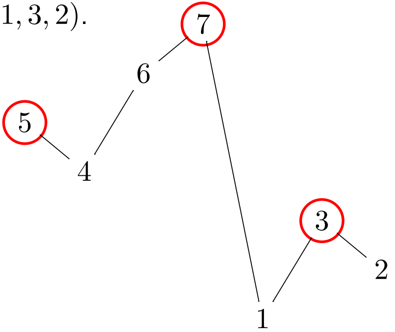

$$\operatorname{des}\left(5, 4, 6, 7, 1, 3, 2\right) = \left\{ 1, 4, 6 \right\}.$$

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- The descent algebra of S_n is

$$\mathcal{D}(S_n) = \mathrm{span}\left\{ X_J : J \subseteq [n-1] \right\} \subset \mathbb{Q}S_n.$$

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  - hyperplane arrangements.

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- The braid arrangement $\mathcal{B}_n$ consists of the following hyperplanes, where $1 \leq i < j \leq n$.

$$H_{ij} = \{\vec{v} \in \mathbb{R}^n : v_i = v_j\} \subset \mathbb{R}^n.$$

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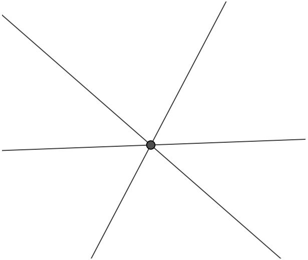

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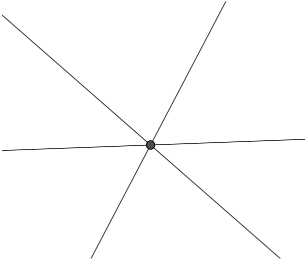

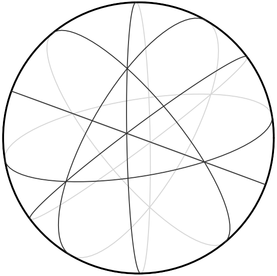

The Faces of $\mathcal{A}$

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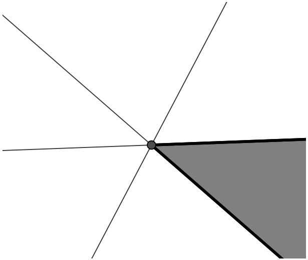
- The connected components of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$ are polyhedra. We call these chambers.



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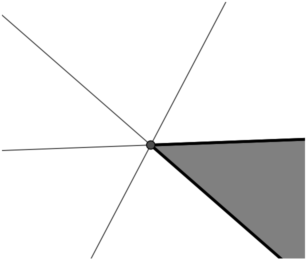

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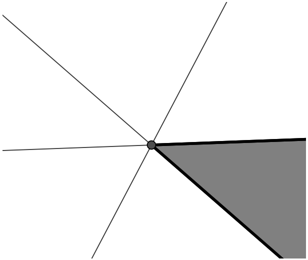

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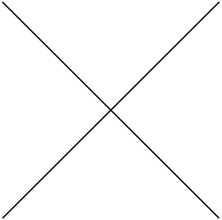
- Note: A chamber is a face!

A Small Example



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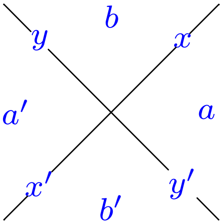

$$\mathcal{F} = \{x, x', y, y', a, a', b, b', \vec{0}\}.$$

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$$\begin{aligned} v_1 = v_5 < v_2 = v_3 < v_4 \longleftrightarrow & (\{1{,}5\}, \{2, 3\}, \{4\}) \\ &= (15, 23, 4). \end{aligned}$$

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where ✂ means "discard empty intersections".

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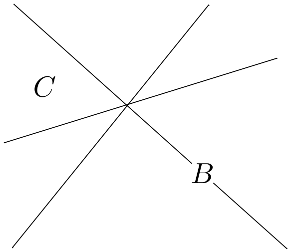

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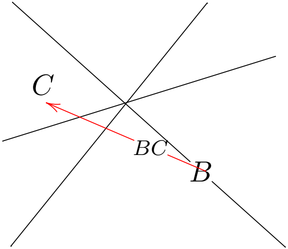

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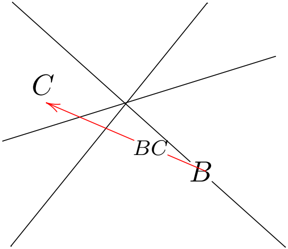

If C is a chamber, then BC is the chamber containing B that is separated from C by the fewest number of hyperplanes.

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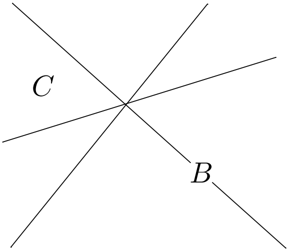

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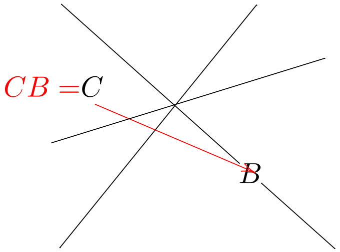

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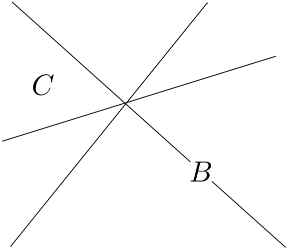

This product was used by Jacques Tits to give another proof that $\mathcal{D}(S_n)$ is an algebra. (In an appendix to Solomon's 1976 paper.)

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- Algebraic techniques give results about the random walks.

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- This action preserves the product.

$$\sigma(BC) = \sigma(B)\sigma(C).$$

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- For example, if $n = 3$,

  $$(1, 2, 3) + (32, 1) + 17(123) - (2, 13).$$

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- Theorem. $(k\mathcal{F})^{S_n}$ is anti-isomorphic to $\mathcal{D}(S_n)$.

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- The composition gives a multplication-reversing algebra homomorphism from $(k\mathcal{F})^{S_n}$ into $(kS_n)^{op}$.

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- Now take the cardinalities of the entries.

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- If G is a group acting on a semigroup X such that $g(xy) = g(x)g(y)$ and $G_x \cap G_y = G_{xy}$, then there is a map from $(RX)^G$ into the ring of characters of G.

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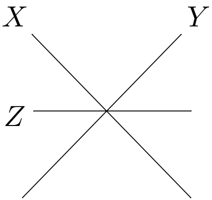

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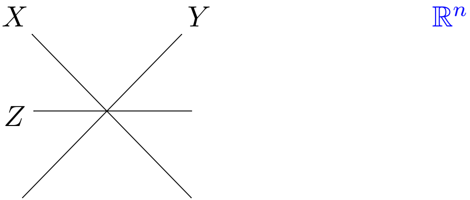

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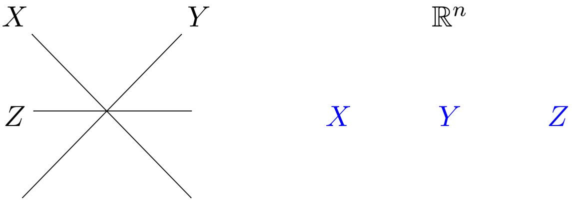

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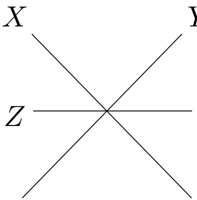

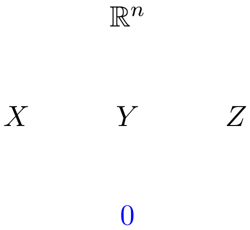

Intersection of at least two hyperplanes.

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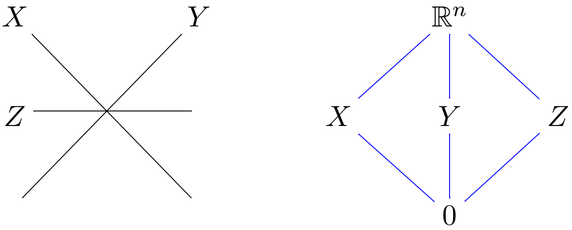

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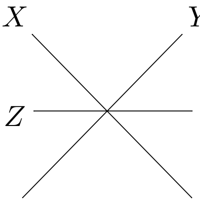

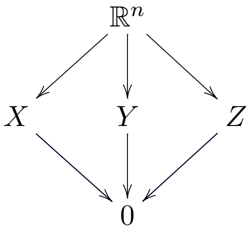

Draw an arrow $X \to Y$ iff X covers Y.

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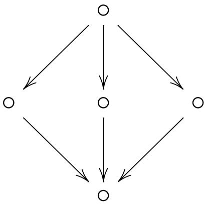

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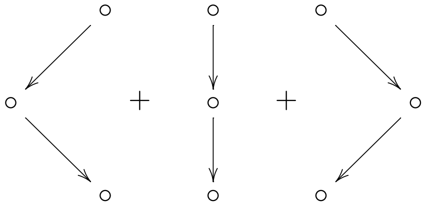

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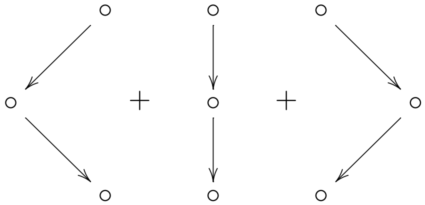

- Corollary. $k\mathcal{F}$ depends only on how the hyperplanes intersect!

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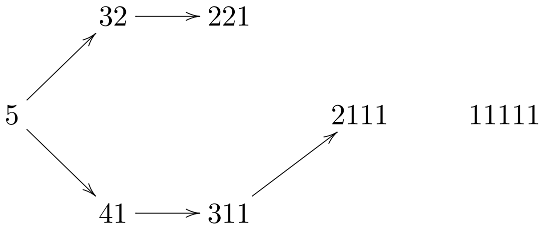

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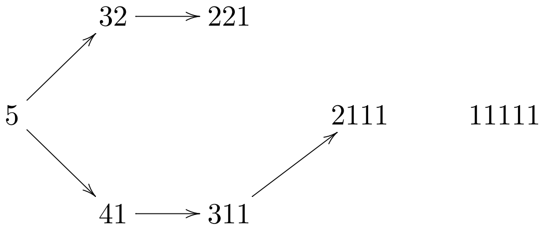

- The kernel of $k\mathcal{Q}_{S_n} \to \mathcal{D}(S_n)$ is not well understood.

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- [S.] There is a nice construction of idempotents in $k\mathcal{F}$. These give idempotents in $\mathcal{D}(S_n)$.

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- This connection is via poset cohomology. The cohomology of the lattice of set partitions of $[n]$ is an S_n-module. Tensoring with the sign representation gives both of the above.

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- Interval Greedoids: a generalization of matroid. Develop an "oriented interval greedoid" theory abstracting the theory of oriented matroids.