Geometry and Algebra of Hyperplane Arrangements

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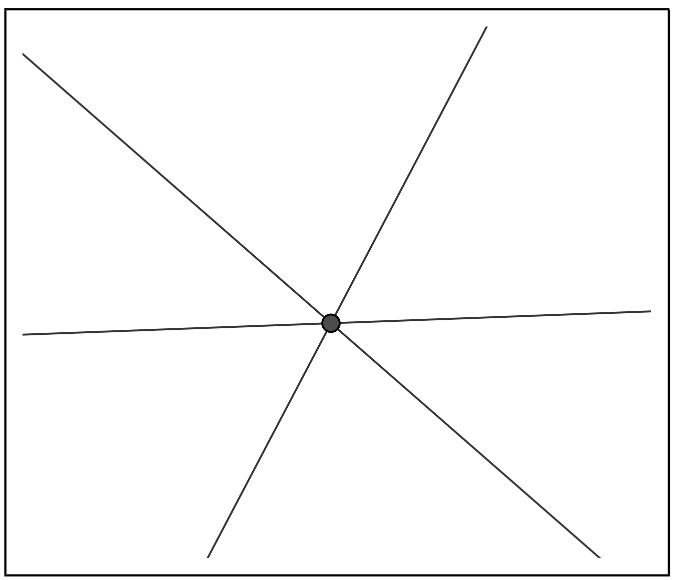
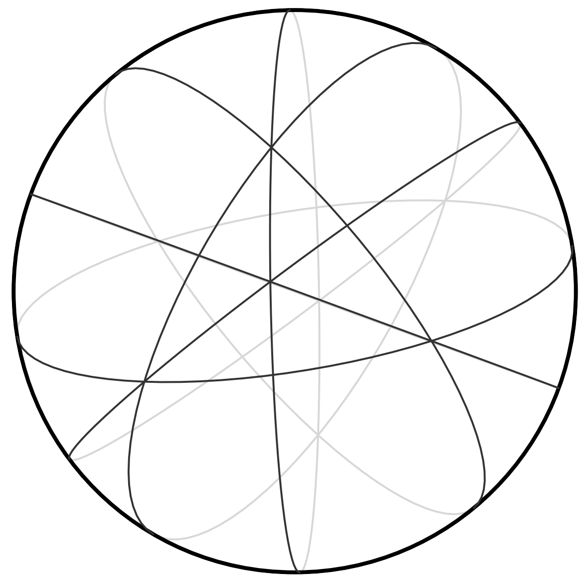
Hyperplane Arrangements

A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes in $\mathbb{R}^n$.

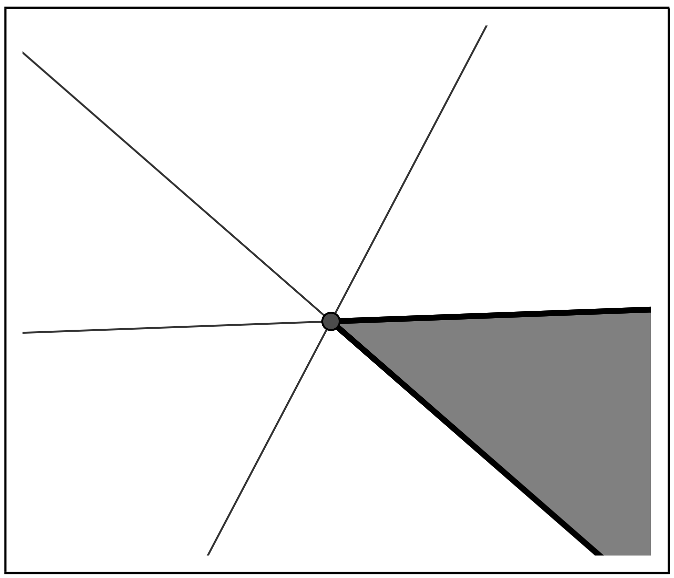


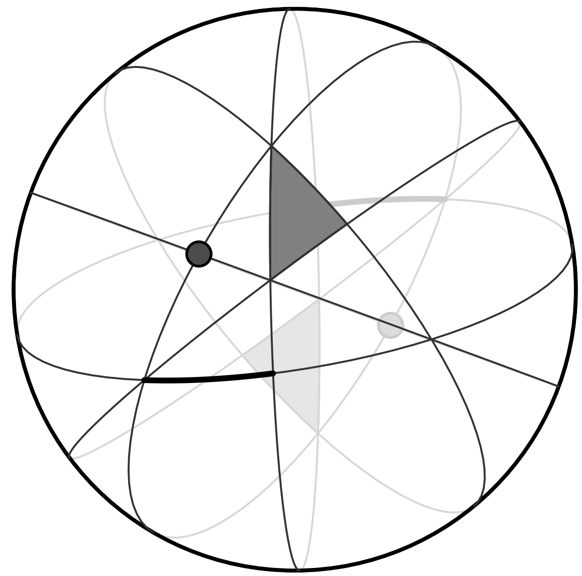


We'll consider *central* hyperplane arrangements: all hyperplanes contain $0 \in \mathbb{R}^n$.

The Faces $\mathcal{F}$

The hyperplanes partition $\mathbb{R}^n$ into polyhedral sets. The set of faces of these polyhedra are the *faces* of $\mathcal{A}$.



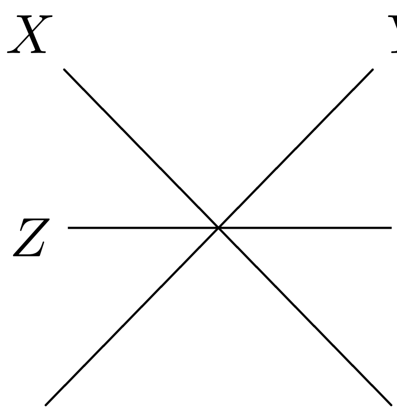


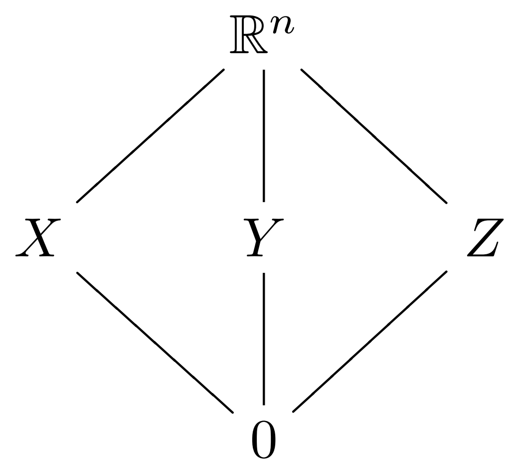
Partial order: $f \leq g \iff f$ is a *face* of g.

The maximal faces are called *chambers*.

The Intersection Lattice $\mathcal{L}$

The *intersection lattice* $\mathcal{L}$ of $\mathcal{A}$ is the collection of all possible intersections of the hyperplanes in $\mathcal{A}$ ordered by *inclusion*.

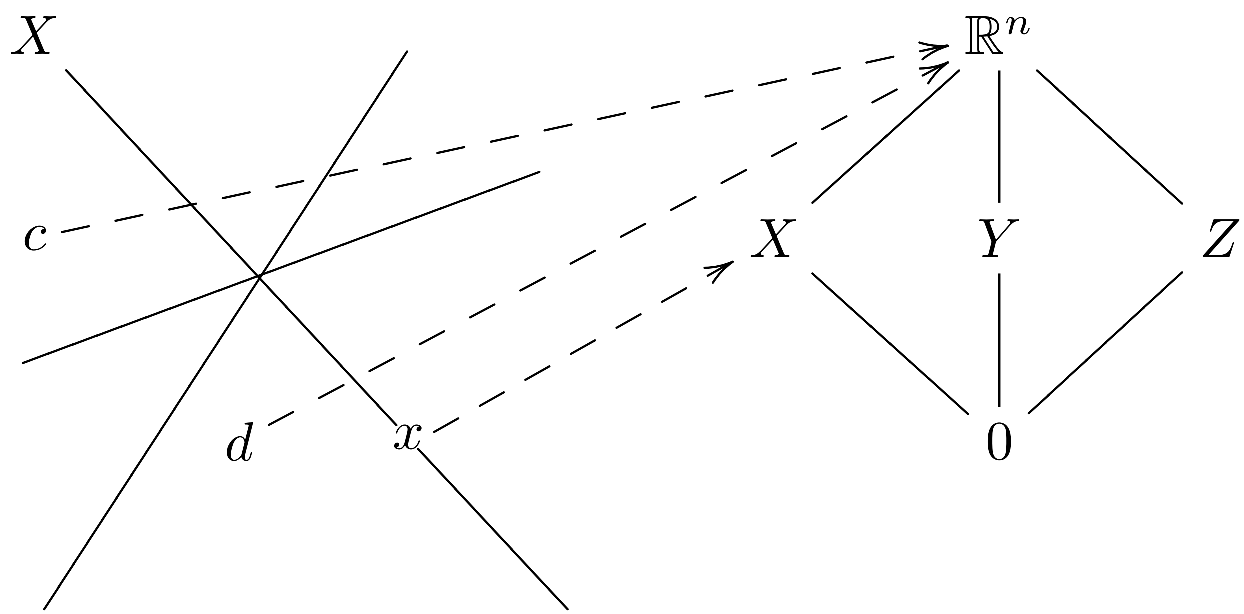




Warning: Others order $\mathcal{L}$ be *reverse* inclusion!

The Support Map

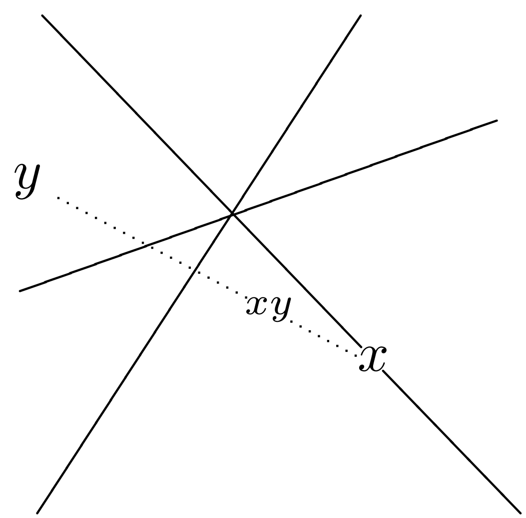
supp : $\mathcal{F} \to \mathcal{L}$ sends a face to the linear span of that face.

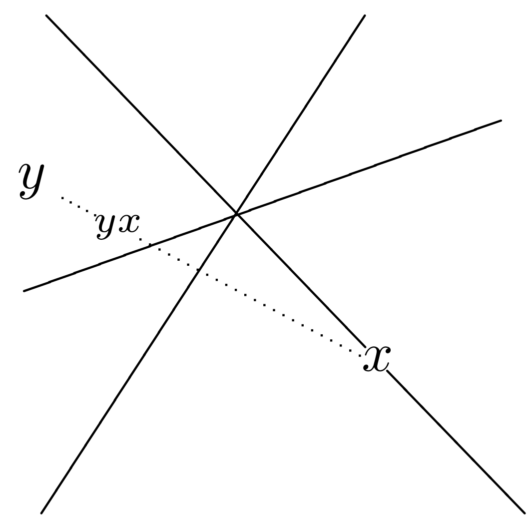


This is an order-preserving surjection of posets.

A Product on $\mathcal{F}$

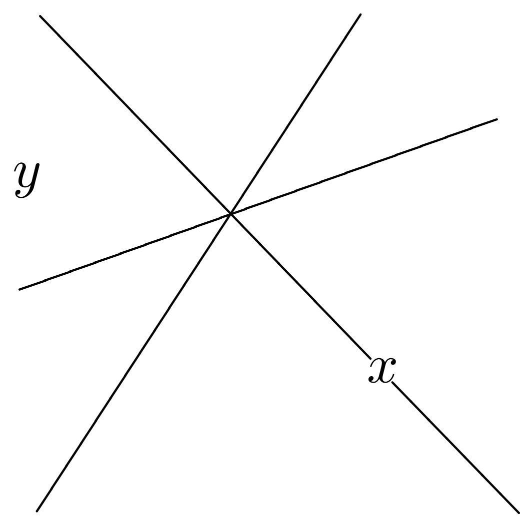
xy is the face you are in by moving a small positive distance along a line from x to y.





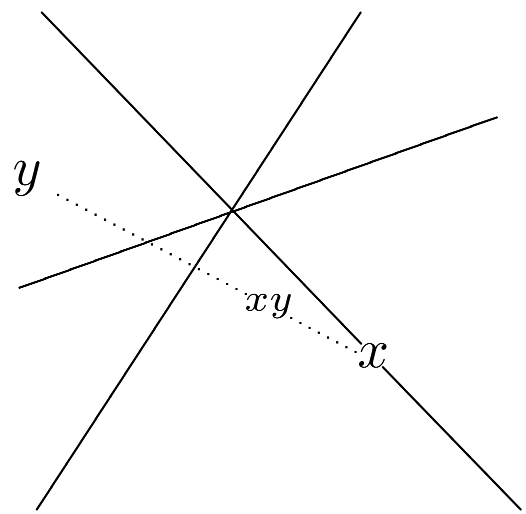
Another charaterization: oriented matroid composition.

Some Computations



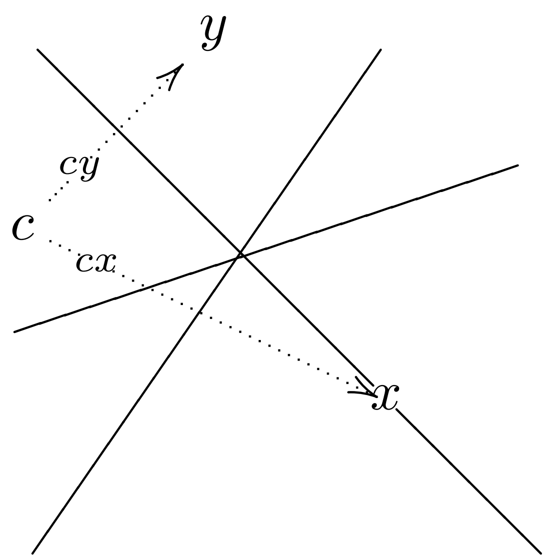
$$x^2 = x$$

Some Computations



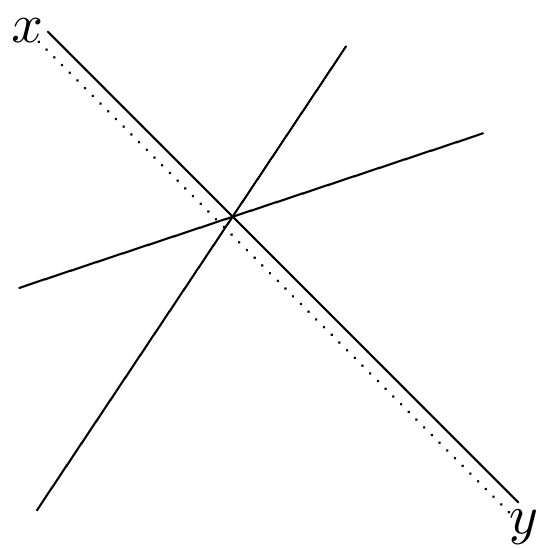
$$xyx = xy$$

Some Computations



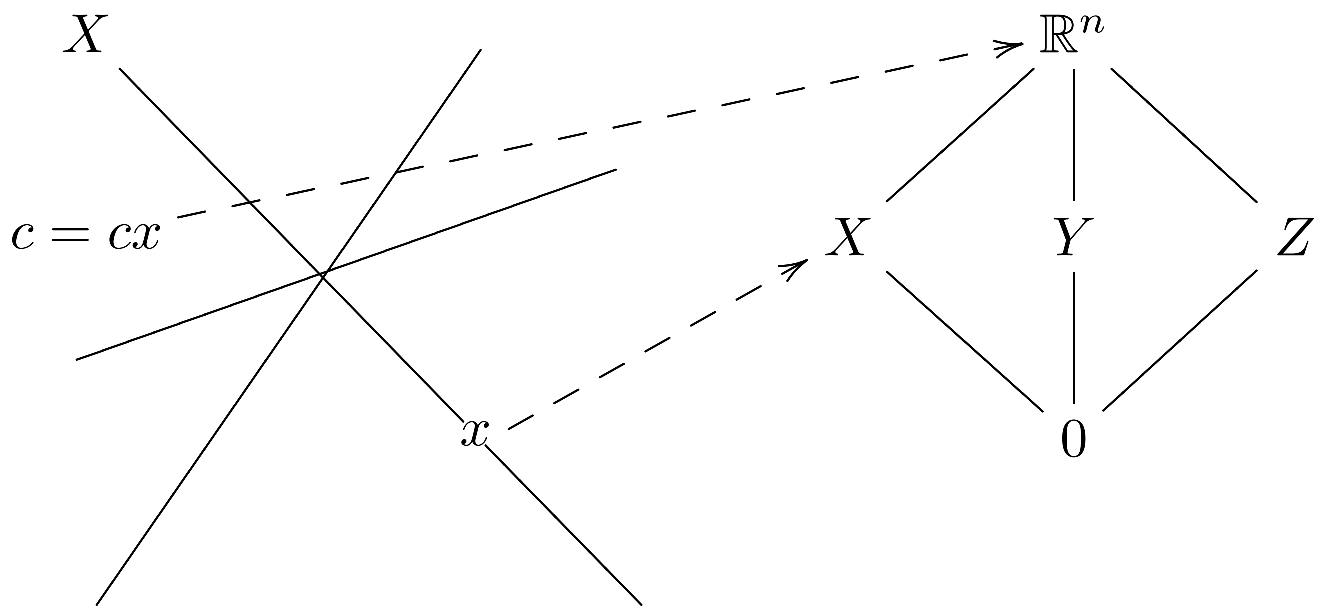
$cx = c$ for all chambers c

Some Computations



$xy = x$ iff $\text{supp}(y) \leq \text{supp}(x)$.

Some Computations



$$\text{supp}(xy) = \text{supp}(x) \vee \text{supp}(y).$$

Therefore, $\text{supp} : \mathcal{F} \to \mathcal{L}$ is a homomorphism of semigroups.

The Face Semigroup Algebra

$k\mathcal{F}$ is the set of formal linear combinations of elements of $\mathcal{F}$

$$\sum_{x \in \mathcal{F}} \lambda_x x$$

with multiplication defined using the product of $\mathcal{F}$.

Markov Chains

» A class of Markov chains can be encoded as random walks on the chambers of $\mathcal{A}$.

» A step in the chain: From c move to xc with probability p_x.

» The transition matrix of the Markov chain is the matrix of the linear transformation of left multiplication by

$$\sum_{x \in \mathcal{F}} p_x x,$$

where p_x is the probability measure on the faces $\mathcal{F}$.

The Descent Algebra

» Let W be a finite Coxeter group. Solomon defined a subalgebra of the group algebra kW called the *descent algebra* of W.

» W is generated by reflections, so the hyperplanes fixed by a reflection of W form a hyperplane arrangement.

» W acts on the faces $\mathcal{F}$, hence also on the semigroup algebra $k\mathcal{F}$.

Theorem: [Bidigare] The invariant subalgebra $(k\mathcal{F})^W$ is isomorphic to Solomon's descent algebra.

Representations of $k\mathcal{F}$

$$\operatorname{supp} : k\mathcal{F} \twoheadrightarrow k\mathcal{L} \cong \prod k$$

kernel is nilpotent

(semisimple)

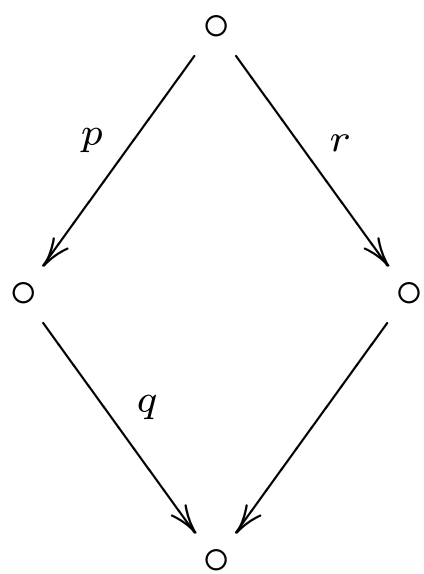
$\Longrightarrow$ Representations are one-dimensional and indexed by $X \in \mathcal{L}$:

$$\chi_X : k\mathcal{F} \to k$$

$$\chi_X(y) = \begin{cases} 1, & \text{if } \operatorname{supp}(y) \leq X, \\ 0, & \text{if } \operatorname{supp}(y) \not\leq X. \end{cases}$$

QUIVERS

A *quiver* Q is a directed graph. The *path algebra* kQ of a quiver is the k-vector space spanned by the paths of Q with multiplication the composition of paths.



$$q \cdot p = qp$$
$$p \cdot q = 0$$
$$p \cdot r = 0$$
$$\circ \cdot \circ = \circ$$

Quivers

If all the simple modules of an algebra A are one-dimensional, then there exists a quiver Q and an algebra surjection $kQ \to A$.

What is the quiver $\mathcal{Q}$ of $k\mathcal{F}$?

(The quiver of an algebra is canonical, but the surjection is not.)

Vertices and Arrows of a Quiver

» The vertices and the arrows of Q generate the path algebra kQ, so they must map to generators of A.

» The vertices map to idempotents in A.

» The arrows, being nilpotent and generators, map to elements in

$$\operatorname{rad}(A)/\operatorname{rad}^2(A).$$

» So the number of arrows from X to Y is

$$\dim_k \left(Y \cdot \operatorname{rad}(A)/\operatorname{rad}^2(A) \cdot X\right).$$

Number of Vertices

Idempotents "correspond" to isomorphism classes of simple modules, so the number of vertices of the quiver is the number of isomorphism classes of irreducible representations.

$$\boxed{\mathcal{Q}_0 = \mathcal{L}.}$$

Number of Arrows

The number of arrows $X \to Y$ is

$$\dim_k \big(Y \cdot \operatorname{rad}(A)/\operatorname{rad}^2(A) \cdot X\big) = \dim_k \operatorname{Ext}^1_{k\mathcal{F}}(k_X, k_Y),$$

where k_X is a simple module corresponding to the vertex X.

I Know What You Are Thinking.

Ext?!

What do I need to know about Ext?

To compute $\mathrm{Ext}^i_{k\mathcal{F}}(k_X, k_Y)$ you need a projective resolution of k_X.

A *projective resolution* is an exact sequence of $k\mathcal{F}$-modules

$$\cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k_X \longrightarrow 0$$

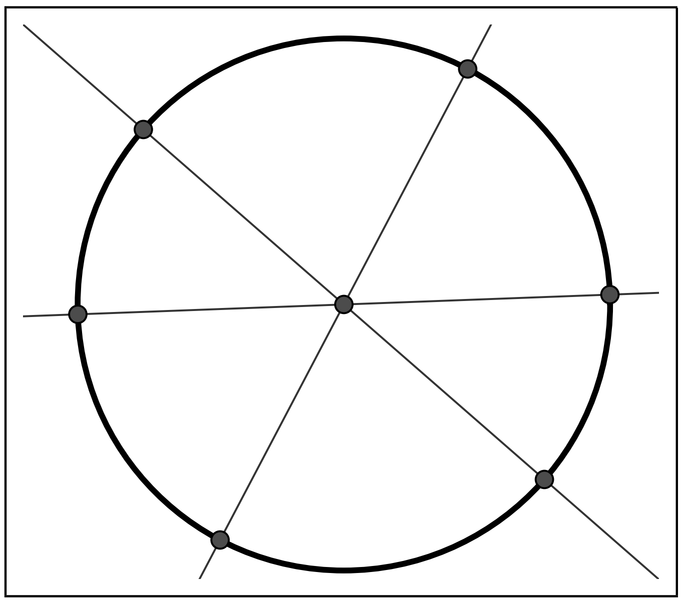
where the modules P_i are *projective* modules.

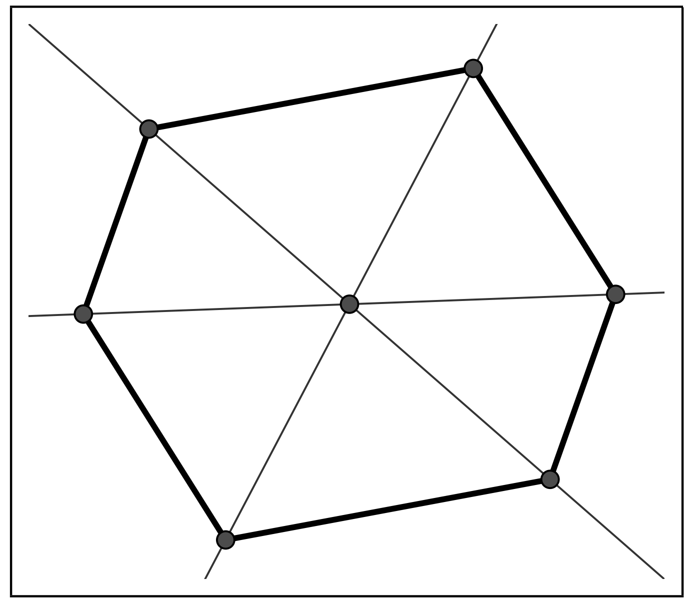
We'll use the *geometry* of the arrangement to construct one.

Projective Resolutions of Simple Modules

» Intersect the arrangement with a sphere centered at the origin.

» The dual cell decomposition of the sphere is a *zonotope* Z.





» The face poset of Z is the opposite of the face poset of $\mathcal{F}$.

» Therefore the augmented cellular chain complex of Z can be identified with the sequence

$$\cdots \xrightarrow{\partial} k\mathcal{F}_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_0 \xrightarrow{\chi} k \longrightarrow 0,$$

where $\mathcal{F}_p$ is the set of codimension p faces in $\mathcal{F}$.

» This sequence is exact since Z has trivial homology.

$$\cdots \xrightarrow{\boxed{\partial}} k\mathcal{F}_p \xrightarrow{\boxed{\partial}} \cdots \xrightarrow{\boxed{\partial}} k\mathcal{F}_0 \xrightarrow{\chi} k \longrightarrow 0$$

The boundary operator ∂ is defined by constructing a set of *incidence numbers* for the faces $\mathcal{F}$.

- Pick an orientation ϵ_X for each subspace $X \in \mathcal{L}$.
- If $x \leq y$, then pick a positively oriented basis $\{\vec{e_1}\}_i$ of $\mathrm{supp}(x)$ and pick a vector $\vec{v}$ in y.
- Define $[x : y] = \epsilon_X(\vec{e_1}, \ldots, \vec{e_r}, \vec{v})$.

$$\partial(x) = \sum_{y \gtrdot x} [x : y] y.$$

$$\cdots \xrightarrow{\partial} \boxed{k\mathcal{F}_p} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \boxed{k\mathcal{F}_0} \xrightarrow{\chi} k \longrightarrow 0$$

Define the action of $\mathcal{F}$ on $\mathcal{F}_p$ by

$$x \cdot y = \begin{cases} xy, & \text{if } \operatorname{supp}(x) \le \operatorname{supp}(y) \\ 0, & \text{if } \operatorname{supp}(x) \not\le \operatorname{supp}(y). \end{cases}$$

Then ∂ is a left $k\mathcal{F}$-module homomoprhism.

$$\cdots \xrightarrow{\partial} k\mathcal{F}_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_0 \xrightarrow{\boxed{\chi}} \boxed{k} \longrightarrow 0$$

For $\chi : k\mathcal{F}_0 \to k$ to be a module morphism, we must have

$$\chi(y) = 1 \text{ for all } y \in \mathcal{F}.$$

Some History

This construction was used by Brown and Diaconis to compute the multiplicities of the eigenvalues of the random walk on the chambers of the hyperplane arrangement.

$$\cdots \xrightarrow{\partial} k\mathcal{F}_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} k\mathcal{F}_0 \xrightarrow{\boxed{\chi}} \boxed{k} \longrightarrow 0$$

Observe that $\chi : k\mathcal{F}_0 \to k$

$$\chi(y) = 1 \text{ for all } y \in \mathcal{F}$$

is the irreducible representation $\chi_{\mathbb{R}^n} : k\mathcal{F} \to k$

$$\chi_{\mathbb{R}^n}(y) = 1 \text{ if } \operatorname{supp}(y) \leq \mathbb{R}^n.$$

So $\boxed{k}$ is the simple module $k_{\mathbb{R}^n}$.

$k\mathcal{F}_p$ IS PROJECTIVE!

$$\cdots \xrightarrow{\partial} \boxed{k\mathcal{F}_p} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \boxed{k\mathcal{F}_0} \xrightarrow{\chi} k \longrightarrow 0$$

Fix $x \in \mathcal{F}$ with support X. Define elements

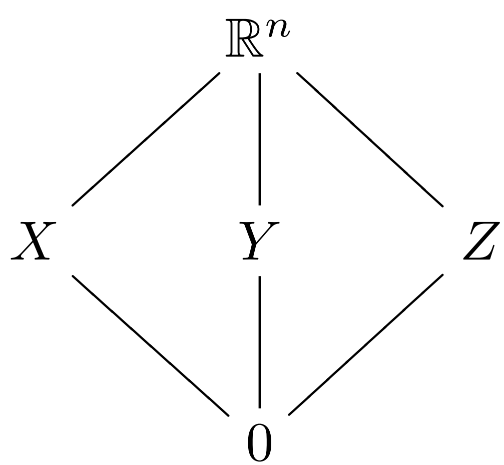
$$e_X = x - \sum_{Y > X} x e_Y .$$

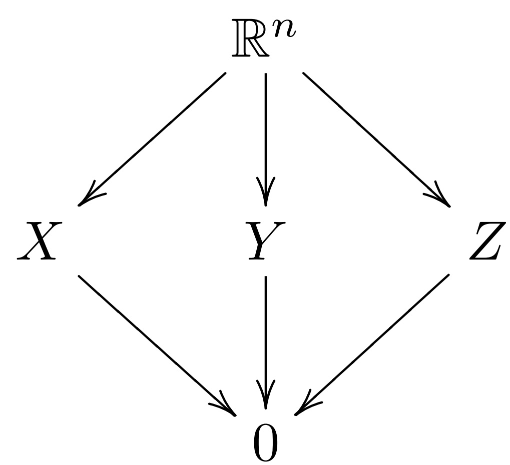
They form a *complete set of primitive orthogonal idempotents* and

$$x e_Y = \begin{cases} x e_Y, & \text{if } \operatorname{supp}(x) \le Y, \\ 0, & \text{if } \operatorname{supp}(x) \not\le Y. \end{cases}$$

This is isomorphic to the action of $\mathcal{F}$ on $\mathcal{F}_p$.

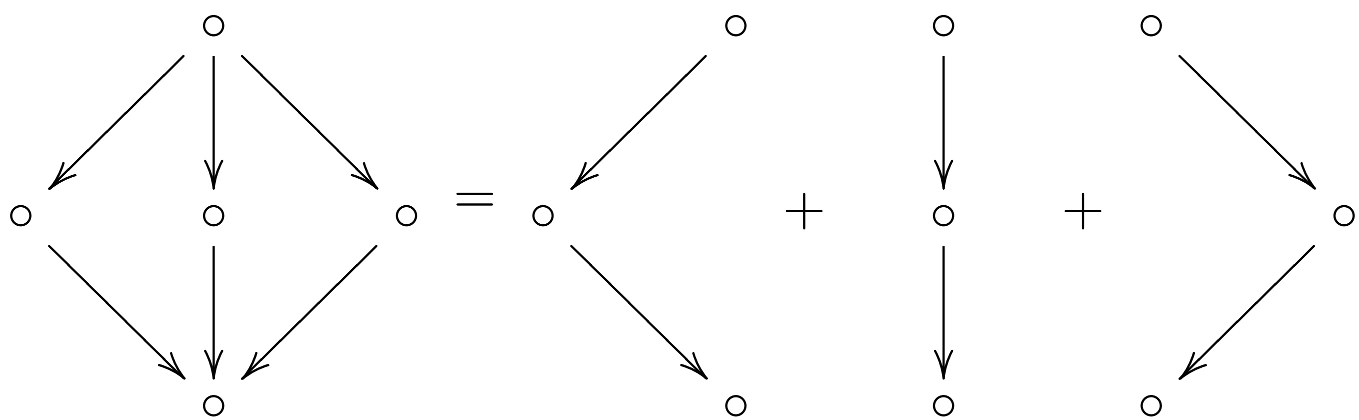
THEOREM: The quiver $\mathcal{Q}$ of $k\mathcal{F}$ coincides with the Hasse diagram of $\mathcal{L}$.





The Relations

For every interval of length two in $\mathcal{L}$ take the sum of all the paths of length that in the interval.



Theorem: Let ρ denote the sum of all paths of length 2 in $\mathcal{Q}$. Then $k\mathcal{F} \cong k\mathcal{Q}/\langle\rho\rangle$.

$k\mathcal{F}$ depends only on $\mathcal{L}$.

Therefore, hyperplane arrangements with isomorphic intersection lattices have isomorphic face semigroup algebras although the face semigroups need not be isomorphic.

$k\mathcal{F}$ is a *Koszul algebra.*

A *Koszul algebra* is an algebra where the simple modules have projective resolutions of "this form".

Therefore,

- The Ext-*algebra* of $k\mathcal{F}$ is the incidence algebra $I(\mathcal{L})$.
- The Ext-*algebra* of $I(\mathcal{L})$ is $k\mathcal{F}$.

Connections with poset cohomology.

» The *poset cohomology* of $\mathcal{L}$ embeds into $k\mathcal{F}$.

» The *poset cohomology* of *every* interval of $\mathcal{L}$ embeds into $k\mathcal{F}$.

» The *Whitney cohomology* of $\mathcal{L}$ embeds into $k\mathcal{F}$.

» A slight modification to the definition of *poset cohomology* gives a cohomology ring with a cup product that is isomorphic to $k\mathcal{F}$.