A *Very* Very Special Class of Algebras

(On a subalgebra of the group algebra of a finite Coxeter group)

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Université du Québec à Montréal

25 May 2007

Quivers from Hyperplane Arrangements

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$\mathbb{R}^3$

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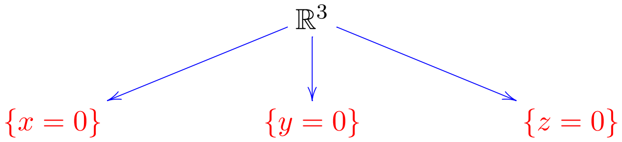

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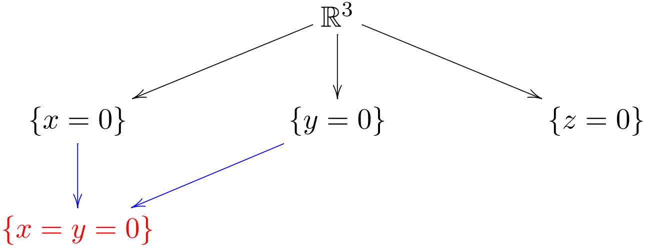

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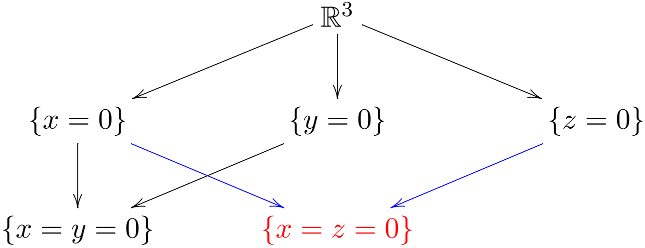

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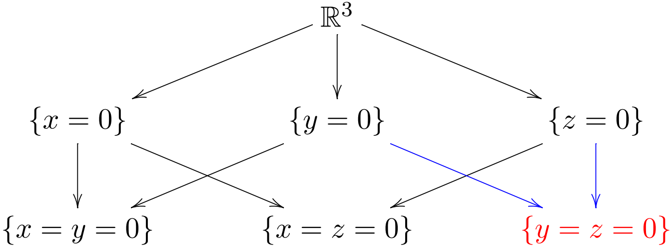

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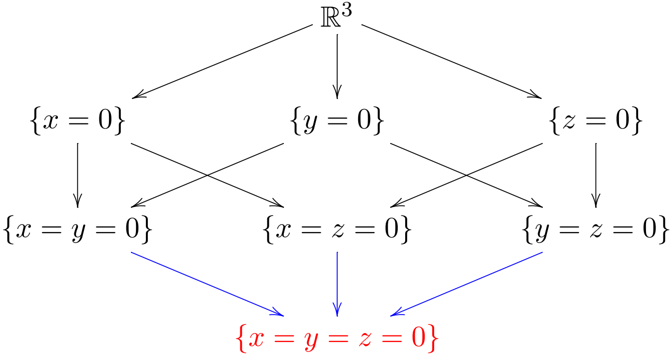

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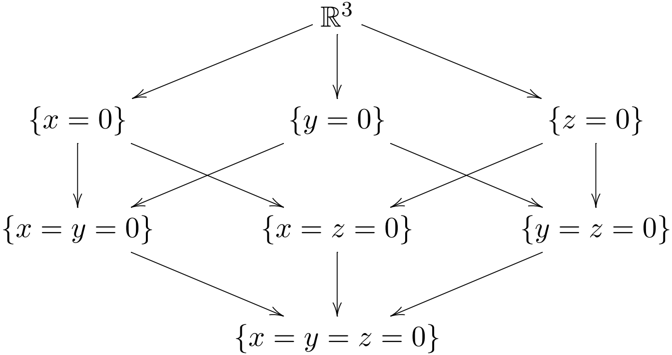

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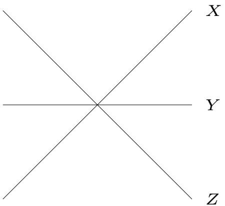

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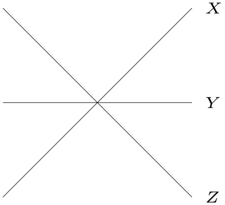

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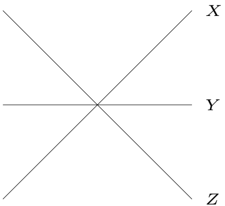

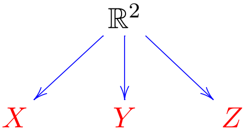

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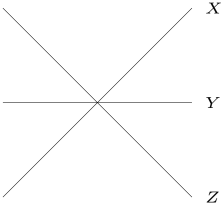

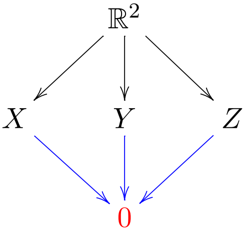

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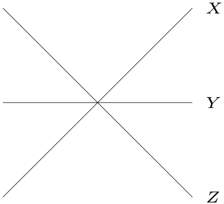

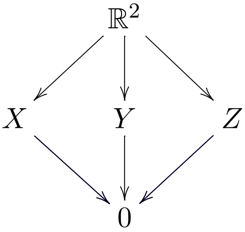

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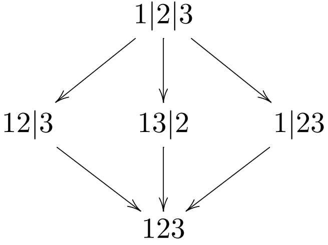
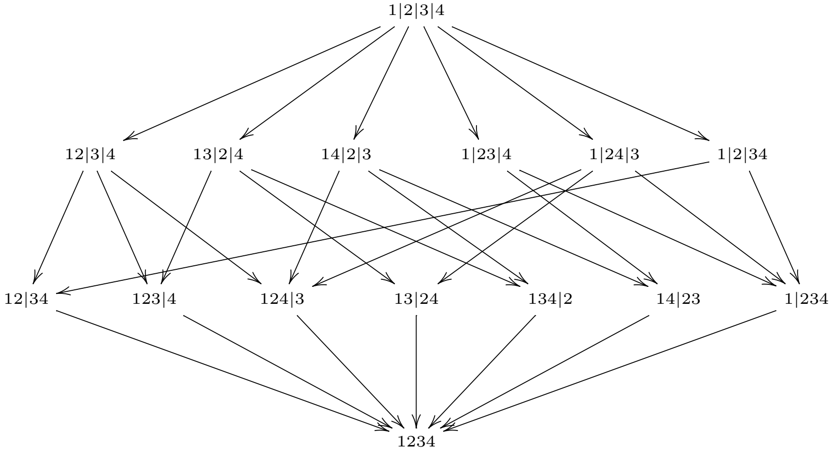
- For $n = 3$:

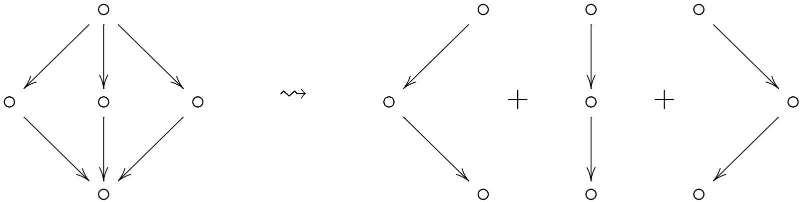


Partition lattice for $n = 4$.



Recall: $\{13|24\} = \{\vec{x} \in \mathbb{R}^4 : x_1 = x_3, x_2 = x_4\}$.

Relations for $\mathcal{Q}$



Every interval of length two gives one relation:
the sum of the paths of length two in the interval.

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- Reason 1: They appear in *nature*.

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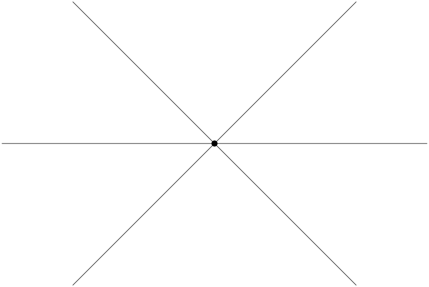
A hyperplane arrangement partitions $\mathbb{R}^n$ into subsets called *faces*.



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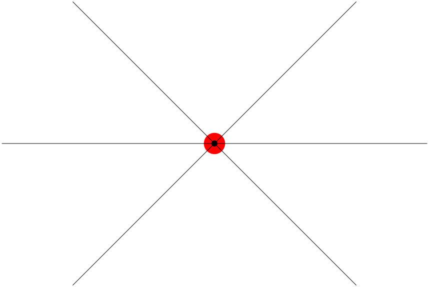

The origin.

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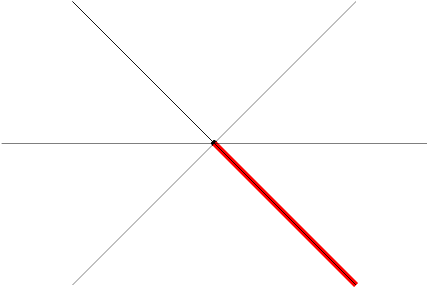

Rays emanating from the origin.

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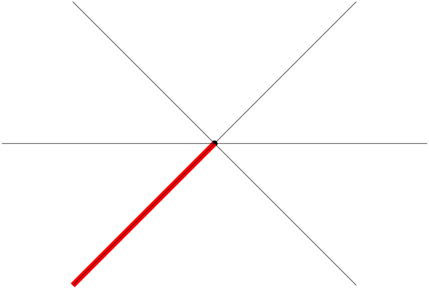

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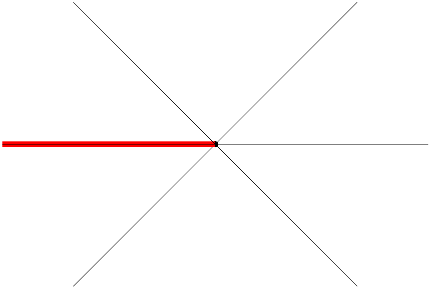

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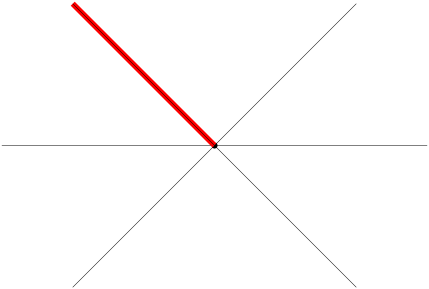

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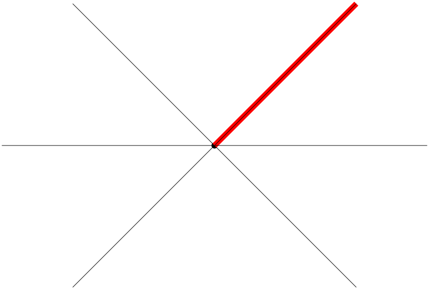

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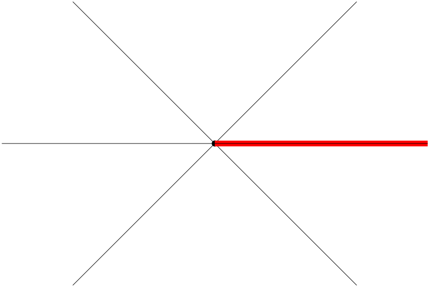

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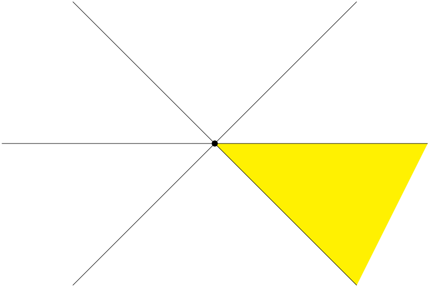

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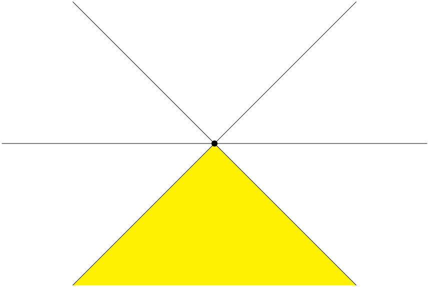

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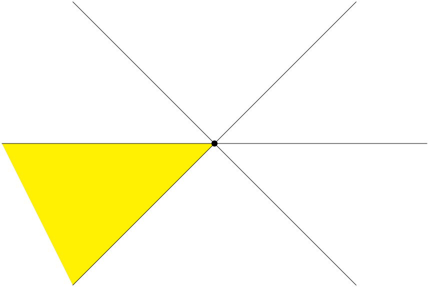

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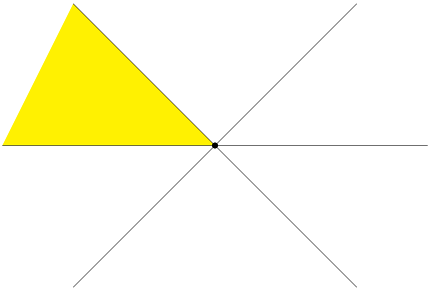

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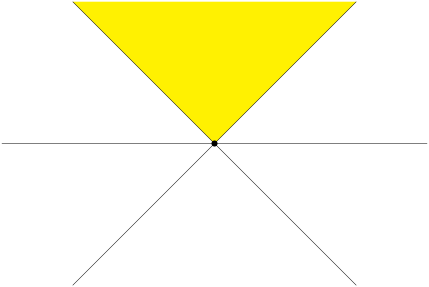

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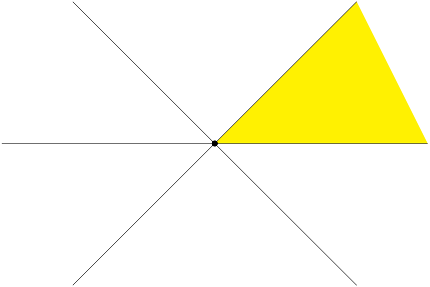

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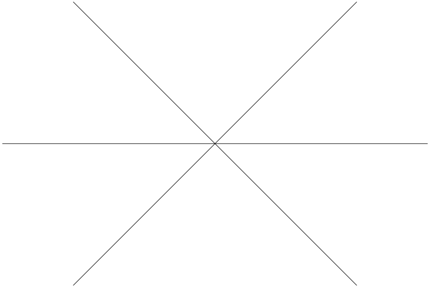

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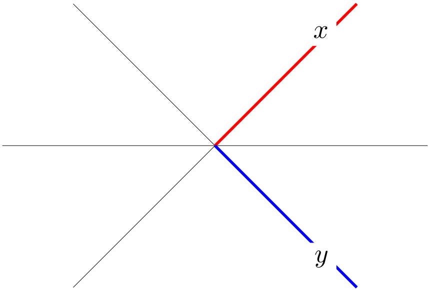

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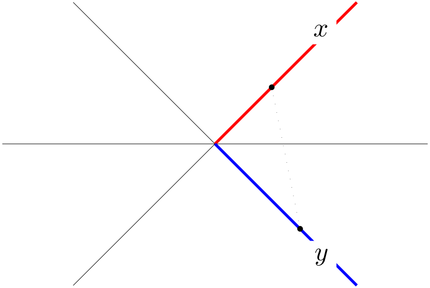

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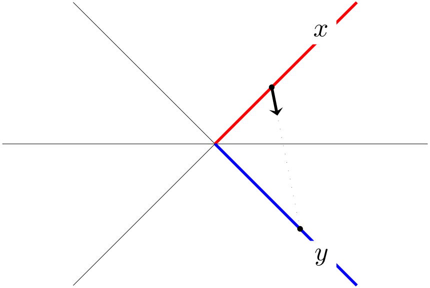

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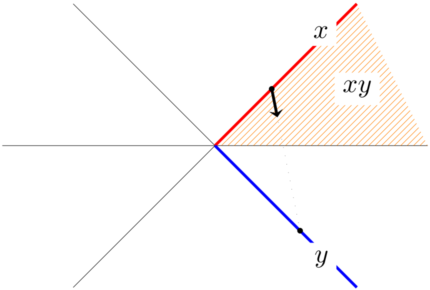

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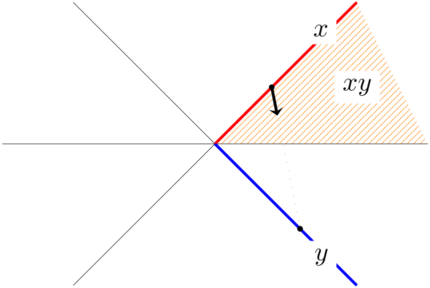

Left Regular Band: an associative semigroup satisfying $x^2 = x$ and $xyx = xy$.

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- *Interest:* product encodes random walks on the regions. Probability $\rightsquigarrow$ algebra!

Connection between $\mathcal{F}$ and $\mathcal{Q}$

Theorem

Let k denote some field and let $\mathcal{F}$ denote the semigroup of faces of a hyperplane arrangement. Then as k-algebras,

$$k\mathcal{F} \cong k\mathcal{Q}/I.$$

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- Reason 2: Every interval gives a quiver in this class.
- Reason 3: $k\mathcal{F}$ is tangible.
  - Simple construction of primitive orthogonal idempotents in $k\mathcal{F}$.
  - Geometry and topology of the arrangement gives minimal projective resolution of simple modules.
  - The Ext spaces of the simple modules can be easily computed.

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  - $q(z) = \sum_{X} z_X^2 - \sum_{Y \lessdot X} z_X z_Y + \sum_{l(Y,X)=2} z_X z_Y.$
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    - "Generic" hyperplane arrangements.
    - Reflection arrangements.

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- *Reflection group W*: a group generated by reflections of $\mathbb{R}^n$.

Example: $W = S_n$ acting on $\mathbb{R}^n$ by permuting coordinates.

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- *Reflection arrangement*: hyperplanes fixed by reflections in W.

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The W action on $\mathbb{R}^n$ induces *two* actions of W on $k\mathcal{Q}$.

Action 1: Since vertices of $\mathcal{Q}$ correspond to intersections of the hyperplanes, W permutes the vertices of $\mathcal{Q}$. This induces an action of W on $\mathcal{Q}$.

Example: For $W = S_n$, the vertices of $\mathcal{Q}$ are set partitions. S_n acts on set partitions element-wise.

$$\tau_{1,3} \cdot \Big(\{146, 5, 23\} \to \{146, 235\} \Big) = \{346, 5, 21\} \to \{346, 215\}\,.$$

Group actions

Action 2: W permutes the hyperplanes in the reflection arrangement; so, W acts on the faces $\mathcal{F}$. This extends to an action of W on $k\mathcal{F} \cong k\mathcal{Q}/I$, which lifts to an action of W on $k\mathcal{Q}$.

Example: S_n action on faces (ordered set partitions):

$$\tau_{1,3} \cdot (7, 23, 56, 14) = (7, 21, 56, 34)\,.$$

On $k\mathcal{Q}$, combine Action 1 with a sign:

$$\omega(X_1 \to \cdots \to X_p) = \\ \mathsf{sign}(\omega, X_1)\mathsf{sign}(\omega, X_p)\Big(\omega(X_1) \to \cdots \to \omega(X_p)\Big).$$

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- *Reason 10:* $(k\mathcal{F})^{S_n}$ is the *descent algebra of S_n*.

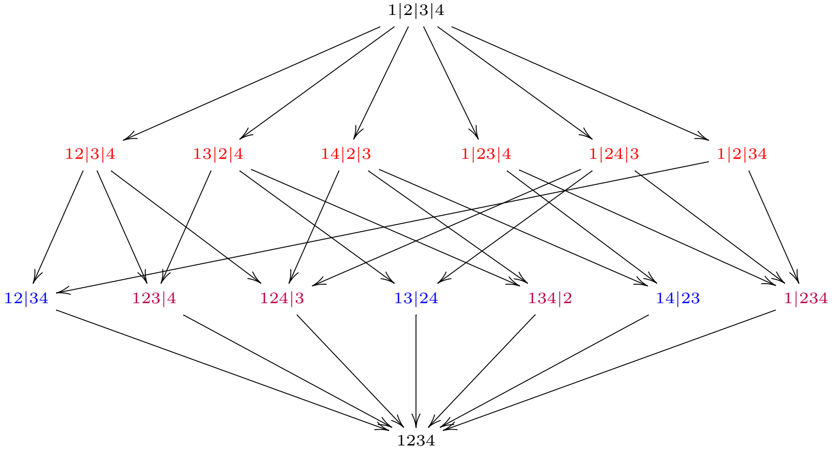
The descent algebra of S_n

- Defined by L. Solomon in 1976 for any finite Coxeter group.
- It enjoys connections with:
  - the representation theory of the symmetric group;
  - the free Lie algebra;
  - probability theory;
  - Hochschild homology of algebras;
  - combinatorics;
  - hyperplane arrangements.

The quiver of $(k\mathcal{F})^{S_n}$

- Determined from the *signed* action of S_n on $k\mathcal{Q}$.
- Lift action of S_n from $k\mathcal{F}$ to $k\mathcal{Q}$ via $k\mathcal{Q} \to k\mathcal{F}$, and consider $(k\mathcal{Q})^{S_n} \to (k\mathcal{Q}/I)^{S_n} \cong (k\mathcal{F})^{S_n}$.
- The vertices are the S_n-orbits of the vertices of $\mathcal{Q}$.
- $[X] \to [Y]$ iff $\nexists \omega \in S_n : \omega(X \to Y) = -(X \to Y)$.

Partition lattice for $n = 4$.



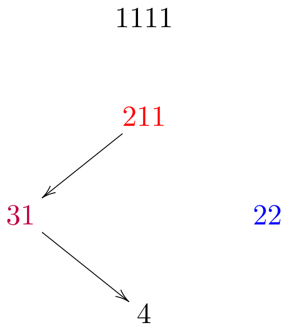
Quiver of $(k\mathcal{F})^{S_n}$

- Vertices are *integer partitions* of n :

$$p_1 \geq p_2 \geq \cdots \geq p_i > 0 \text{ with } \sum p_i = n.$$

- $p \longrightarrow q$ iff q is obtained from p by merging two distinct parts of p.

- For $n = 4$:



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5. Explicit descriptions of the Tits and Euler forms.
6. Hochschild cohomology is $\mathrm{HH}^i(k\mathcal{F}) = 0$ for $i > 0$.
7. $k\mathcal{F}$ is $\mathbb{Z}$-graded and graded by a lattice.
8. There are interesting subclasses.
9. Reflection arrangements $\rightsquigarrow$ group actions!
10. $(k\mathcal{F})^W$ is the descent algebra of W.

Some questions

Suppose G is a group acting on an algebra A.

- What can be said about A^G?
- For $G = S_n$ and $A = k\mathcal{F}$ we have, for all $p \geq 0$,

$$\mathrm{rad}^p(A^G) = \mathrm{rad}^p(A) \cap A^G.$$

  How often does this hold? Other reflection groups?
- How do you find the quiver of A^G knowing the quiver of A?
- Given the quiver, when can one find relations for A^G?
- What about $A * G$?
- Gröbner bases?
- What does Koszul give you? The Koszul dual is very nice!
- Left regular bands: which give Koszul algebras?

One last thing. . .

I will be looking for a job in the Fall.