Permutations, Card Shuffling, and Representation Theory

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Permutations and Increasing Subsequences

Permutations : Definitions and Notations

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Permutation : bijection $\sigma : \{1, 2, \ldots, n\} \to \{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\}$

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symmetric group $\mathfrak{S}_n$: group of permutations of $\{1, \ldots, n\}$

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

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Example

σ	123					
increasing subsequences of length 2						
$\mathrm{inc}_2(\sigma)$						

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123					
increasing subsequences of length 2	12					
$\mathrm{inc}_2(\sigma)$						

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123					
increasing subsequences of length 2	12 1 3					
$\mathrm{inc}_2(\sigma)$						

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Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123					
increasing subsequences of length 2	12 1 3 23					
$\mathrm{inc}_2(\sigma)$						

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$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123					
increasing subsequences of length 2	12 1 3 23					
$\mathrm{inc}_2(\sigma)$	3					

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123	132				
increasing subsequences of length 2	12 1 3 23	13				
$\mathrm{inc}_2(\sigma)$	3					

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Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123	132				
increasing subsequences of length 2	12 1 3 23	13 1 2				
$\mathrm{inc}_2(\sigma)$	3					

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$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123	132			
increasing subsequences of length 2	12 1 3 23	13 1 2			
$\mathrm{inc}_2(\sigma)$	3	2			

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123	132	213			
increasing subsequences of length 2	12 1 3 23	13 1 2	2 3 13			
$\mathrm{inc}_2(\sigma)$	3	2	2			

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123	132	213	231		
increasing subsequences of length 2	12 1 3 23	13 1 2	2 3 13	23		
$\mathrm{inc}_2(\sigma)$	3	2	2	1		

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123	132	213	231	312
increasing subsequences of length 2	12 1 3 23	13 1 2	2 3 13	23	12
$\mathrm{inc}_2(\sigma)$	3	2	2	1	1

increasing subsequences

Definition

$$\mathrm{inc}_k(\sigma) = \#\left\{ \begin{array}{c} \text{increasing subsequences} \\ \text{of length } k \text{ in } \sigma \end{array} \right\}$$

Example

σ	123	132	213	231	312	321
increasing subsequences of length 2	12, 1 3, 23	13, 1 2	2 3, 13	23	12	
$\mathrm{inc}_2(\sigma)$	3	2	2	1	1	0

Matrix of increasing k-subsequences

$$\mathrm{Inc}_{n,k} = \begin{matrix} & \\ \sigma \end{matrix} \begin{pmatrix} & \tau & \\ & \vdots & \\ \cdots & \mathrm{inc}_k(\tau^{-1}\sigma) & \cdots \\ & \vdots & \end{pmatrix}$$

$$\mathrm{Inc}_{3,1} = \left[\mathrm{inc}_1(\tau^{-1}\sigma)\right]$$

	123	132	213	231	312	321
123						
132						
213						
231						
312						
321						

$$\mathrm{Inc}_{3,1} = \left[\mathrm{inc}_1(\tau^{-1}\sigma) \right]$$

$$\begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array}
\begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix} \end{array}$$

$$\mathrm{Inc}_{3,2} = \left[\mathrm{inc}_2(\tau^{-1}\sigma) \right]$$

	123	132	213	231	312	321
123						
132						
213						
231						
312						
321						

$$\mathrm{Inc}_{3,2} = \left[\mathrm{inc}_2(\tau^{-1}\sigma) \right]$$

	123	132	213	231	312	321
123	3					
132	2					
213	2					
231	1					
312	1					
321	0					

$$\mathrm{Inc}_{3,2} = \Big[\mathrm{inc}_2(\tau^{-1}\sigma) \Big]$$

	123	132	213	231	312	321
123	3	2				
132	2	3				
213	2	1				
231	1	0				
312	1	2				
321	0	1				

$$\mathrm{Inc}_{3,2} = \left[\mathrm{inc}_2(\tau^{-1}\sigma) \right]$$

	123	132	213	231	312	321
123	3	2	2	1	1	0
132	2	3	1	0	2	1
213	2	1	3	2	0	1
231	1	0	2	3	1	2
312	1	2	0	1	3	2
321	0	1	1	2	2	3

$$\mathrm{Inc}_{3,3} = \left[\mathrm{inc}_3(\tau^{-1}\sigma) \right]$$

$$\begin{array}{c|cccccc} & 123 & 132 & 213 & 231 & 312 & 321 \\ \hline 123 & 1 & 0 & 0 & 0 & 0 & 0 \\ 132 & 0 & 1 & 0 & 0 & 0 & 0 \\ 213 & 0 & 0 & 1 & 0 & 0 & 0 \\ 231 & 0 & 0 & 0 & 1 & 0 & 0 \\ 312 & 0 & 0 & 0 & 0 & 1 & 0 \\ 321 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$\mathrm{Inc}_{4,1} = \left[\mathrm{inc}_1(\tau^{-1}\sigma) \right]$$

$$\begin{pmatrix}
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4
\end{pmatrix}$$

$$\mathrm{Inc}_{4,2} = \left[\mathrm{inc}_2(\tau^{-1}\sigma)\right]$$

$$\begin{pmatrix}
6&5&5&4&4&3&5&4&4&3&3&2&4&3&3&2&2&1&3&2&2&1&1&0\\
5&6&4&3&5&4&4&5&3&2&4&3&3&2&2&1&1&0&4&3&3&2&2&1\\
5&4&6&5&3&4&4&3&3&2&2&1&5&4&4&3&3&2&2&3&1&0&2&1\\
4&3&5&6&4&5&3&2&2&1&1&0&4&5&3&2&4&3&3&4&2&1&3&2\\
4&5&3&4&6&5&3&4&2&1&3&2&2&3&1&0&2&1&5&4&4&3&3&2\\
3&4&4&5&5&6&2&3&1&0&2&1&3&4&2&1&3&2&4&5&3&2&4&3\\
5&4&4&3&3&2&6&5&5&4&4&3&3&2&4&3&1&2&2&1&3&2&0&1\\
4&5&3&2&4&3&5&6&4&3&5&4&2&1&3&2&0&1&3&2&4&3&1&2\\
4&3&3&2&2&1&5&4&6&5&3&4&4&3&5&4&2&3&1&0&2&3&1&2\\
3&2&2&1&1&0&4&3&5&6&4&5&3&2&4&5&3&4&2&1&3&4&2&3\\
3&4&2&1&3&2&4&5&3&4&6&5&1&0&2&3&1&2&4&3&5&4&2&3\\
2&3&1&0&2&1&3&4&4&5&5&6&2&1&3&4&2&3&3&2&4&5&3&4\\
4&3&5&4&2&3&3&2&4&3&1&2&6&5&5&4&4&3&1&2&0&1&3&2\\
3&2&4&5&3&4&2&1&3&2&0&1&5&6&4&3&5&4&2&3&1&2&4&3\\
3&2&4&3&1&2&4&3&5&4&2&3&5&4&6&5&3&4&0&1&1&2&2&3\\
2&1&3&2&0&1&3&2&4&5&3&4&4&3&5&6&4&5&1&2&2&3&3&4\\
2&1&3&4&2&3&1&0&2&3&1&2&4&5&3&4&6&5&3&4&2&3&5&4\\
1&0&2&3&1&2&2&1&3&4&2&3&3&4&4&5&5&6&2&3&3&4&4&5\\
3&4&2&3&5&4&2&3&1&2&4&3&1&2&0&1&3&2&6&5&5&4&4&3\\
2&3&3&4&4&5&1&2&0&1&3&2&2&3&1&2&4&3&5&6&4&3&5&4\\
2&3&1&2&4&3&3&4&2&3&5&4&0&1&1&2&2&3&5&4&6&5&3&4\\
1&2&0&1&3&2&2&3&3&4&4&5&1&2&2&3&3&4&4&3&5&6&4&5\\
1&2&2&3&3&4&0&1&1&2&2&3&3&4&2&3&5&4&4&5&3&4&6&5\\
0&1&1&2&2&3&1&2&2&3&3&4&2&3&3&4&4&5&3&4&4&5&5&6
\end{pmatrix}$$

$$\mathrm{Inc}_{4,3} = \left[\mathrm{inc}_3(\tau^{-1}\sigma) \right]$$

$$\begin{pmatrix}
4&2&2&1&1&0&2&0&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0\\
2&4&1&0&2&1&0&2&0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0\\
2&1&4&2&0&1&1&0&0&0&0&0&2&0&1&1&0&0&0&1&0&0&0&0\\
1&0&2&4&1&2&1&0&0&0&0&0&0&2&0&0&1&1&0&1&0&0&0&0\\
1&2&0&1&4&2&0&1&0&0&0&0&0&1&0&0&0&0&2&0&1&1&0&0\\
0&1&1&2&2&4&0&1&0&0&0&0&0&1&0&0&0&0&0&2&0&0&1&1\\
2&0&1&1&0&0&4&2&2&1&1&0&0&0&1&0&0&0&0&0&1&0&0&0\\
0&2&0&0&1&1&2&4&1&0&2&1&0&0&1&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&2&1&4&2&0&1&1&1&2&0&0&0&0&0&0&1&0&0\\
1&0&0&0&0&0&1&0&2&4&1&2&0&0&0&2&1&1&0&0&0&1&0&0\\
0&1&0&0&0&0&1&2&0&1&4&2&0&0&0&1&0&0&1&1&2&0&0&0\\
0&1&0&0&0&0&0&1&1&2&2&4&0&0&0&1&0&0&0&0&0&2&1&1\\
1&1&2&0&0&0&0&0&1&0&0&0&4&2&2&1&1&0&0&0&0&0&1&0\\
0&0&0&2&1&1&0&0&1&0&0&0&2&4&1&0&2&1&0&0&0&0&1&0\\
0&0&1&0&0&0&1&1&2&0&0&0&2&1&4&2&0&1&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&2&1&1&1&0&2&4&1&2&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&1&0&0&1&2&0&1&4&2&1&1&0&0&2&0\\
0&0&0&1&0&0&0&0&0&1&0&0&0&1&1&2&2&4&0&0&1&1&0&2\\
1&1&0&0&2&0&0&0&0&0&1&0&0&0&0&0&1&0&4&2&2&1&1&0\\
0&0&1&1&0&2&0&0&0&0&1&0&0&0&0&0&1&0&2&4&1&0&2&1\\
0&0&0&0&1&0&1&1&0&0&2&0&0&0&0&0&0&1&2&1&4&2&0&1\\
0&0&0&0&1&0&0&0&1&1&0&2&0&0&0&0&0&1&1&0&2&4&1&2\\
0&0&0&0&0&1&0&0&0&0&0&1&1&1&0&0&2&0&1&2&0&1&4&2\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&0&2&0&1&1&2&2&4
\end{pmatrix}$$

$$\mathrm{Inc}_{4,4} = \left[\mathrm{inc}_4(\tau^{-1}\sigma) \right]$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

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- for each n, we have a family of n matrices

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each of which is $n! \times n!$.

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Motivation : Mysteries

- for each n, we have a family of n matrices

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- *Questions* : Is this true ? Why ? What are these integers ?

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they do commute !

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connections with probability and representation theory :

- card shuffling and related random walks
- representation theory of the symmetric group

Card Shuffling

random-to-random shuffle

deck of cards :

$$\sigma_1\,\sigma_2\,\sigma_3\,\sigma_4\,\sigma_5\,\sigma_6\,\sigma_7\,\sigma_8\,\sigma_9$$

random-to-random shuffle

deck of cards :

$$\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5\ \sigma_6\ \sigma_7\ \sigma_8\ \sigma_9$$

remove a card at random :

$$\sigma_3$$

$$\sigma_1\ \sigma_2\ \uparrow\ \sigma_4\ \sigma_5\ \sigma_6\ \sigma_7\ \sigma_8\ \sigma_9$$

random-to-random shuffle

deck of cards :

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remove a card at random :

$$\sigma_1\ \sigma_2\ \overset{\sigma_3}{\uparrow}\ \sigma_4\ \sigma_5\ \sigma_6\ \sigma_7\ \sigma_8\ \sigma_9$$

insert the card at random :

$$\sigma_1\ \sigma_2\ \sigma_4\ \sigma_5\ \sigma_6\ \sigma_7\ \overset{\downarrow}{\sigma_3}\ \sigma_8\ \sigma_9$$

Transition matrix of the random-to-random shuffle

entries : probability of going from σ to τ using one shuffle

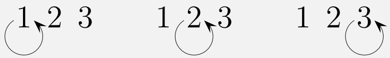
$$\begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \left(\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \right) \end{array}$$

Transition matrix of the random-to-random shuffle

entries : probability of going from σ to τ using one shuffle

	123	132	213	231	312	321
123	$\frac{3}{9}$					
132						
213						
231						
312						
321						

3 ways to obtain **123** from **123** :



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132						
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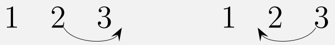
2 ways to obtain **132** from **123** :



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123	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{2}{9}$			
132						
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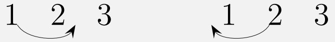

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	123	132	213	231	312	321
123	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$		
132						
213						
231						
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321						

1 way to obtain **231** from **123** :

1 2 3

Transition matrix of the random-to-random shuffle

entries : probability of going from σ to τ using one shuffle

	123	132	213	231	312	321
123	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	
132						
213						
231						
312						
321						

1 way to obtain **312** from **123** :

$1 \quad 2 \quad 3$

Transition matrix of the random-to-random shuffle

entries : probability of going from σ to τ using one shuffle

$$
\begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array}
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\begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\
\begin{pmatrix}
\frac{3}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} & \frac{0}{9} \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & &
\end{pmatrix}
\end{array}
$$

Transition matrix of the random-to-random shuffle

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\begin{pmatrix}
3 & 2 & 2 & 1 & 1 & 0 \\
2 & 3 & 1 & 0 & 2 & 1 \\
2 & 1 & 3 & 2 & 0 & 1 \\
1 & 0 & 2 & 3 & 1 & 2 \\
1 & 2 & 0 & 1 & 3 & 2 \\
0 & 1 & 1 & 2 & 2 & 3
\end{pmatrix}
\end{array}
\times \frac{1}{9}
$$

Transition matrix of the random-to-random shuffle

entries : probability of going from σ to τ using one shuffle

	123	132	213	231	312	321
123	3	2	2	1	1	0
132	2	3	1	0	2	1
213	2	1	3	2	0	1
231	1	0	2	3	1	2
312	1	2	0	1	3	2
321	0	1	1	2	2	3

$\times \frac{1}{9}$

$\mathrm{Inc}_{n,n-1} \overset{\text{(renorm.)}}{=}$ random-to-random shuffle

Properties of the transition matrix

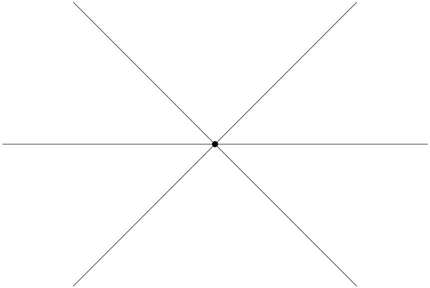
The transition matrix T governs properties of the random walk.

typical questions	$\longleftrightarrow$	**algebraic properties**
probability after m steps	$\longleftrightarrow$	entries of T^m
long-term behaviour (limiting distribution)	$\longleftrightarrow$	eigenvectors $\vec{\pi}$ s.t. $\vec{\pi}\,\mathsf{T} = \vec{\pi}$
rate of convergence $\vec{v}\,\mathsf{T}^n \longrightarrow \vec{\pi}$	$\longleftrightarrow$	governed by eigenvalues of T

random walks on the chambers of a hyperplane arrangement

faces of a hyperplane arrangement

a set of hyperplanes partitions $\mathbb{R}^n$ into *faces* :



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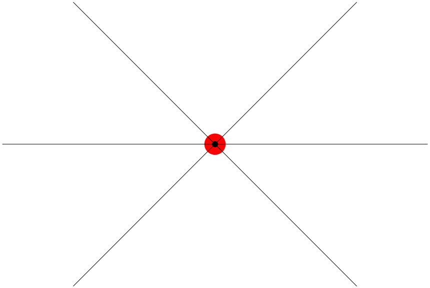

the origin

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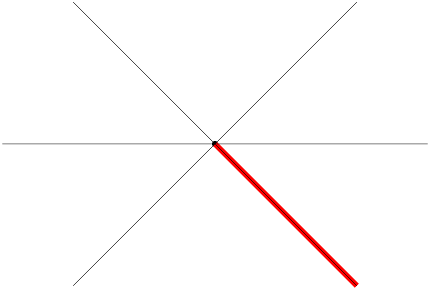

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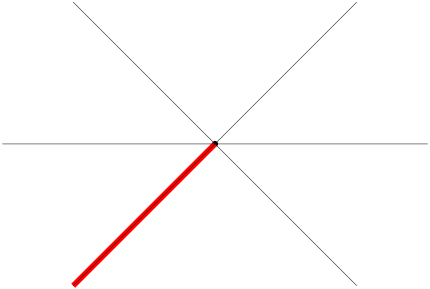

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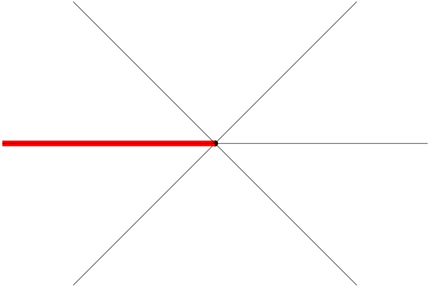

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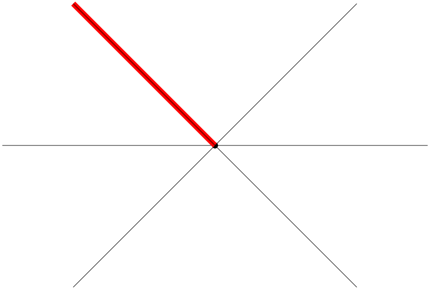

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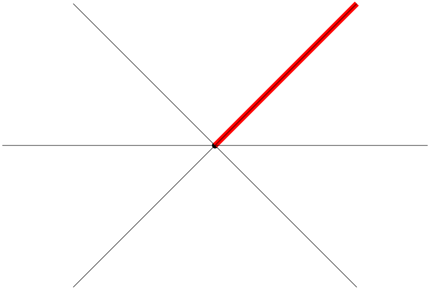

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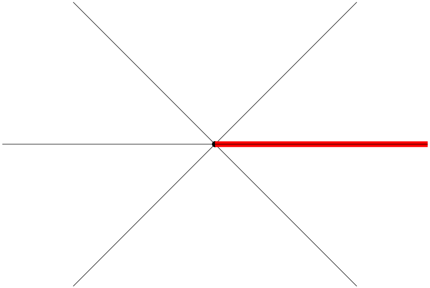

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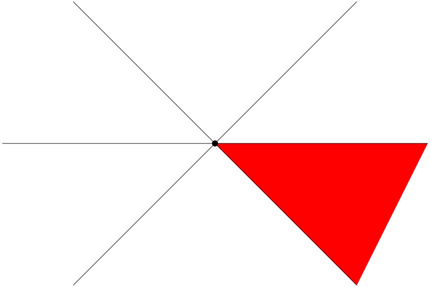

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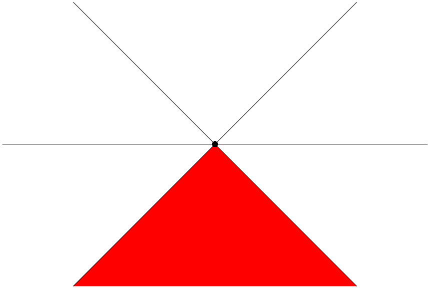

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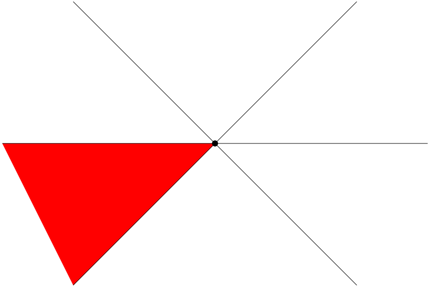

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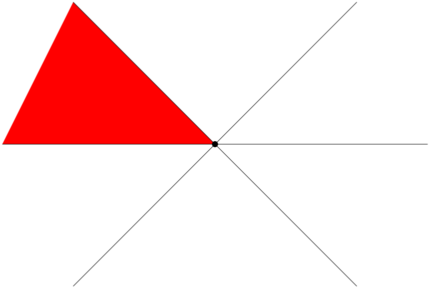

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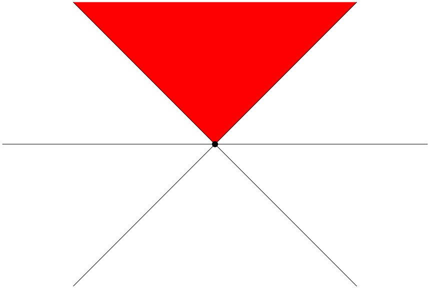

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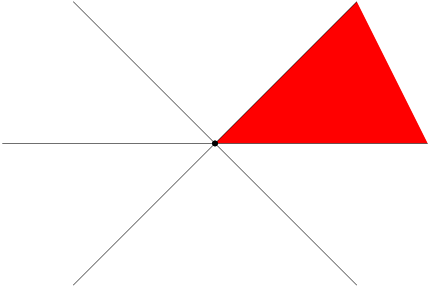
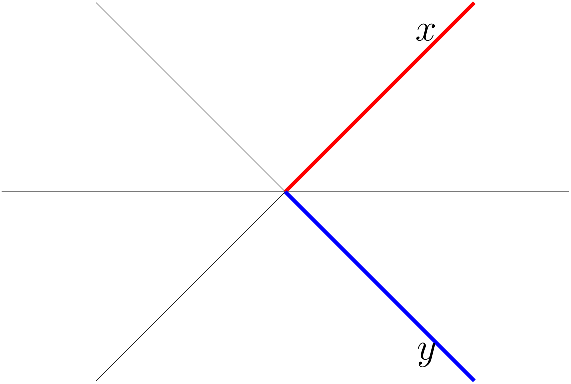


chambers cut out by the hyperplanes

product of two faces

$$xy := \begin{cases} \text{the face first encountered after a small} \\ \text{movement along a line from } x \text{ toward } y \end{cases}$$



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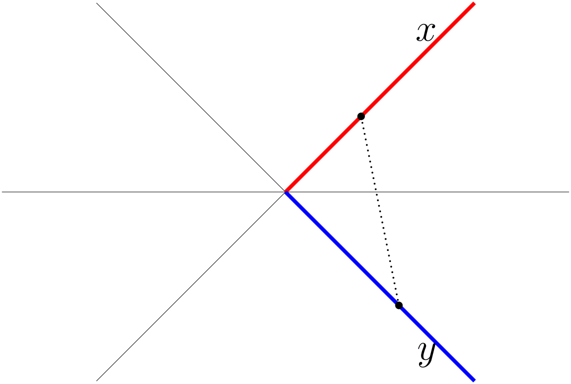

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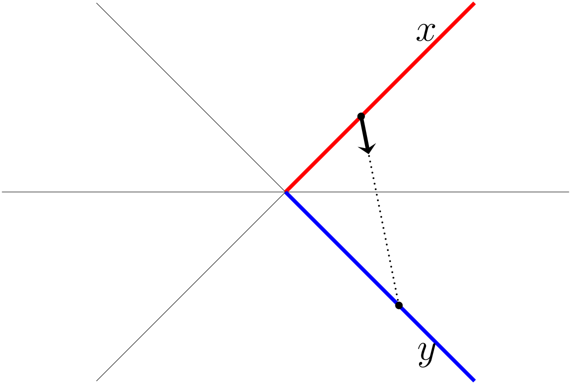

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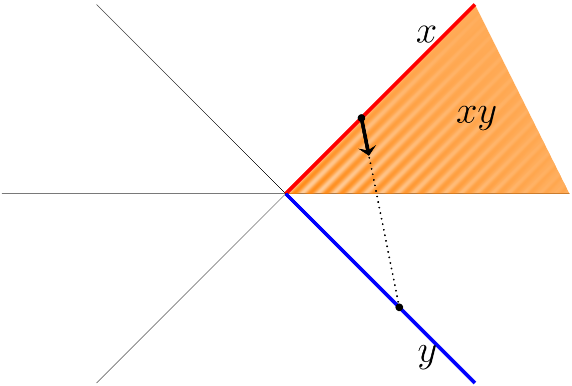

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- if $\vec{v}$ lies on $H_{i,j}$, then $v_i < v_j$ becomes $v_i = v_j$:

$$v_1 = v_5 < v_2 = v_3 < v_4 \longleftrightarrow [\{1, 5\}, \{2, 3\}, \{4\}]$$

Special Case : The "Braid" Arrangement

combinatorial description :

faces $\leftrightarrow$ *ordered* set partitions of $\{1, \ldots, n\}$:

$$[\{2,3\},\{4\},\{1,5\}] \neq [\{4\},\{1,5\},\{2,3\}]$$

chambers $\leftrightarrow$ partitions into singletons

$$[\{2\},\{3\},\{4\},\{1\},\{5\}]$$

product $\leftrightarrow$ intersection of sets in the partition

Product of set compositions

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$$\Big[\{2,5\}\{1,3,4,6\}\Big]\cdot\Big[\{4\}\{1\}\{5\}\{6\}\{3\}\{2\}\Big]$$

Product of set compositions

$$\left[\underline{\overline{\{2,5\}}}\{1,3,4,6\}\right]\cdot\left[\underline{\overline{\{4\}}}\{1\}\{5\}\{6\}\{3\}\{2\}\right]$$

$$=\left[\{2,5\}\cap\{4\}\right.$$

Product of set compositions

$$\left[\underline{\overline{\{2,5\}}}\{1,3,4,6\}\right]\cdot\left[\underline{\overline{\{4\}}}\{1\}\{5\}\{6\}\{3\}\{2\}\right]$$

$$=\left[\emptyset\right.$$

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Product of set compositions

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$$=\Big[\{5\}\{2\}\{1,3,4,6\}\cap\{4\}$$

Product of set compositions

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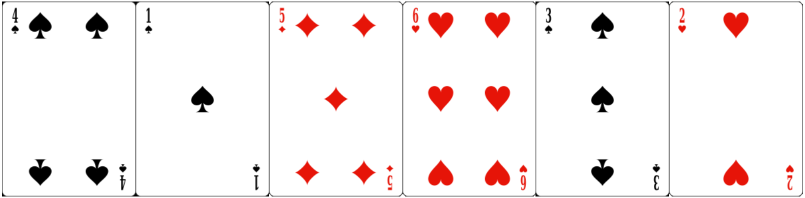

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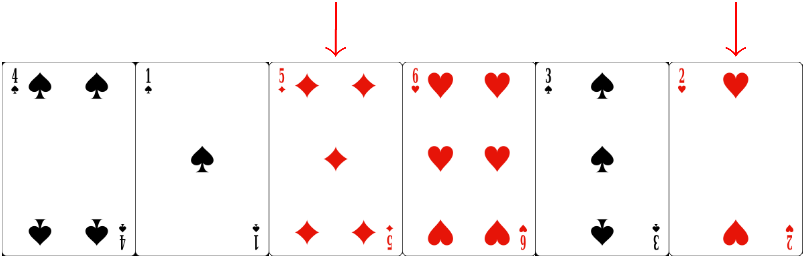
$$\left[{\color{red}\{2,5\}}\{1,3,4,6\}\right]\cdot\left[\{4\}\{1\}\{5\}\{6\}\{3\}\{2\}\right]$$
$$=\left[\{5\}\{2\}\{4\}\{1\}\{6\}\{3\}\right]$$



Product of set compositions

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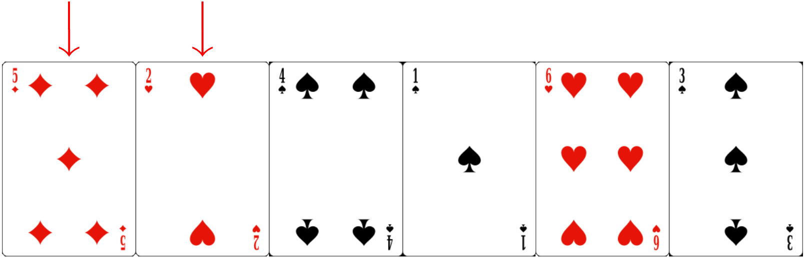

Random walks on hyperplane arrangements

A step in the random walk :

starting from an element $\boldsymbol{c}$,
pick an element $\boldsymbol{y}$ *at random*,
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(Inverse) Riffle Shuffle and Random-to-Top

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Others :

Björner, Athanasiadis-Diaconis, Chung-Graham, . . .

$\mathrm{Inc}_{n,k}$ and hyperplane chamber walks

Theorem (Reiner–S–Welker 2011)

$$\mathrm{Inc}_{n,k} = \mathsf{T}_k \, \mathsf{T}_k^t$$

where T_k is the transition matrix of a random walk on the braid arrangement.

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Consequently, $\ker(\mathrm{Inc}_{n,k}) = \ker(\mathsf{T}_k)$

Representation theory

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Examples :

- trivial representation :

$$\rho(g) = [1]$$

Regular representation of $\mathfrak{S}_3$

$$\rho(\mathbf{123}) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad \rho(\mathbf{132}) = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

$$\rho(\mathbf{213}) = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix} \qquad \rho(\mathbf{231}) = \begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

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$$3\rho(\mathbf{123}) + 2\rho(\mathbf{132}) + 2\rho(\mathbf{213}) + 1\rho(\mathbf{231}) + 1\rho(\mathbf{312}) = \begin{bmatrix} 3 & 2 & 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 & 0 & 1 \\ 1 & 0 & 2 & 3 & 1 & 2 \\ 1 & 2 & 0 & 1 & 3 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 \end{bmatrix}$$

$\mathrm{Inc}_{n,k}$ and the regular representation of $\mathfrak{S}_n$

Theorem (Reiner–S–Welker 2011)

$$\mathrm{Inc}_{n,k} = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{inc}_k(\sigma)\, \mathrm{reg}(\sigma)$$

where reg is the regular representation of $\mathfrak{S}_n$.

Calculation of eigenvalues via irreducibles

(decompose the regular representation into irreducible representations)

$3 \cdot \mathbf{123} \quad 2 \cdot \mathbf{132} \quad 2 \cdot \mathbf{213} \quad 1 \cdot \mathbf{231} \quad 1 \cdot \mathbf{312}$

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trivial :

$$3 \cdot [1] + 2 \cdot [1] + 2 \cdot [1] + 1 \cdot [1] + 1 \cdot [1]$$

$$= [9]$$

Calculation of eigenvalues via irreducibles

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$$3 \cdot \mathbf{123} \quad 2 \cdot \mathbf{132} \quad 2 \cdot \mathbf{213} \quad 1 \cdot \mathbf{231} \quad 1 \cdot \mathbf{312}$$

sign :

$$3 \cdot [1] + 2 \cdot [-1] + 2 \cdot [-1] + 1 \cdot [1] + 1 \cdot [1]$$

$$= [1]$$

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$\chi :$

$$3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}$$

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$$\mathrm{eigs}(\mathrm{Inc}_{3,2}) = \{9, 1, 4, 4, 0, 0\}$$

Final Remarks

- This research was facilitated by computer exploration with the math. software Sage (`sagemath.org`) and Sage-Combinat :
    - to test the conjectures
    - to compute the eigenvalues of the matrices
    - to construct irreducible representations of $\mathfrak{S}_n$
    - to decompose the eigenspaces into irreducible representations
    - to search for other transition matrices with these properties
    - to provide ideas for proofs
- Second family of random walks with similar properties. Our analysis combines what we've seen today :
    - BHR theory of random walks to analyze the kernels ; and
    - representation theory to analyze the other eigenspaces.