Worksheet 6 - Linear Algebra

35 seconds ago by admin

Linear Algebra

Vectors

To create a vector in Sage, use the **vector** command.

Exercise: Create the vector $x = (1, 2, \ldots, 100)$.

Note: vectors in Sage are *row* vectors!

Exercise: Create the vector $y = (1^2, 2^2, \ldots, 100^2)$.

Exercise: Type **x.** and hit tab to see the available methods for vectors. Find the *norm* (length) of the vectors **x** and **y**.

Exercise: Find the *dot product* of **x** and **y**.

[The above problems are essentially the first problem on Exercise Set 1 of William Stein's Math 480b.]

Matrices

Exercise: Use the **matrix** command to create the following matrix over the rational numbers (hint: in Sage, **QQ** denotes the field of rational numbers).

$$\begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 & 0 & 1 \\ 1 & 0 & 2 & 3 & 1 & 2 \\ 1 & 2 & 0 & 1 & 3 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 \end{pmatrix}$$

1. Find the *echelon form* of the above matrix.
2. Find the *right kernel* of the matrix.
3. Find the *eigenvalues* of the matrix.
4. Find the *left eigenvectors* of the matrix.
5. Find the *right eigenspaces* of the matrix.

Exercise: For what values of k is the *determinant* of the following matrix 0?

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{pmatrix}$$

[K. R. Matthews, Elementary Linear Algebra, Chapter 4, Problem 8]

Exercise: Prove that the determinant of the following matrix is -8.

$$\begin{pmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{pmatrix}$$

[K. R. Matthews, *Elementary Linear Algebra*, Chapter 4, Problem 3]

Exercise: Prove that if $a \neq c$, then the line through the points (a, b) and (c, d) is given by the following equation.

$$\det \begin{pmatrix} x & y & 1 \\ a & b & 1 \\ c & d & 1 \end{pmatrix} = 0.$$

Exercise: Find the determinant of the following matrices.

$$\left(1\right), \begin{pmatrix} 1 & 1 \\ r & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ r & 1 & 1 \\ r & r & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 & 1 \\ r & r & 1 & 1 & 1 \\ r & r & r & 1 & 1 \\ r & r & r & r & 1 \end{pmatrix}$$

Make a conjecture about the determinant of an arbitrary matrix in this sequence. Can you prove it your conjecture?

[Adapted from: K. R. Matthews, *Elementary Linear Algebra*, Chapter 4, Problem 19]

Exercise: What is the largest determinant possible for a 3×3 matrix whose entries are $1, 2, \ldots, 9$ (each occurring exactly once, in any order). How many matrices M achieve this maximum?

(*Hint:* You might find the command **Permutations** useful. The following code will construct all the lists that have the entries $1, 2, 3, 4$, each appearing exactly once.)

```
for P in Permutations(4):
    L = list(P)
    print L
```

```
for P in Permutations(4):
   L = list(P)
   print L
```

```
[1, 2, 3, 4]
[1, 2, 4, 3]
[1, 3, 2, 4]
[1, 3, 4, 2]
[1, 4, 2, 3]
[1, 4, 3, 2]
[2, 1, 3, 4]
[2, 1, 4, 3]
[2, 3, 1, 4]
[2, 3, 4, 1]
[2, 4, 1, 3]
[2, 4, 3, 1]
[3, 1, 2, 4]
[3, 1, 4, 2]
[3, 2, 1, 4]
[3, 2, 4, 1]
[3, 4, 1, 2]
[3, 4, 2, 1]
[4, 1, 2, 3]
[4, 1, 3, 2]
[4, 2, 1, 3]
[4, 2, 3, 1]
[4, 3, 1, 2]
[4, 3, 2, 1]
```

Project Euler Problem 11

In the 20×20 grid below, four numbers along a diagonal line have been

marked in red.

```
08 02 22 97 38 15 00 40 00 75 04 05 07 78 52 12 50 77 91 08
49 49 99 40 17 81 18 57 60 87 17 40 98 43 69 48 04 56 62 00
81 49 31 73 55 79 14 29 93 71 40 67 53 88 30 03 49 13 36 65
52 70 95 23 04 60 11 42 69 24 68 56 01 32 56 71 37 02 36 91
22 31 16 71 51 67 63 89 41 92 36 54 22 40 40 28 66 33 13 80
24 47 32 60 99 03 45 02 44 75 33 53 78 36 84 20 35 17 12 50
32 98 81 28 64 23 67 10 26 38 40 67 59 54 70 66 18 38 64 70
67 26 20 68 02 62 12 20 95 63 94 39 63 08 40 91 66 49 94 21
24 55 58 05 66 73 99 26 97 17 78 78 96 83 14 88 34 89 63 72
21 36 23 09 75 00 76 44 20 45 35 14 00 61 33 97 34 31 33 95
78 17 53 28 22 75 31 67 15 94 03 80 04 62 16 14 09 53 56 92
16 39 05 42 96 35 31 47 55 58 88 24 00 17 54 24 36 29 85 57
86 56 00 48 35 71 89 07 05 44 44 37 44 60 21 58 51 54 17 58
19 80 81 68 05 94 47 69 28 73 92 13 86 52 17 77 04 89 55 40
04 52 08 83 97 35 99 16 07 97 57 32 16 26 26 79 33 27 98 66
88 36 68 87 57 62 20 72 03 46 33 67 46 55 12 32 63 93 53 69
04 42 16 73 38 25 39 11 24 94 72 18 08 46 29 32 40 62 76 36
20 69 36 41 72 30 23 88 34 62 99 69 82 67 59 85 74 04 36 16
20 73 35 29 78 31 90 01 74 31 49 71 48 86 81 16 23 57 05 54
01 70 54 71 83 51 54 69 16 92 33 48 61 43 52 01 89 19 67 48
```

The product of these numbers is $26 \times 63 \times 78 \times 14 = 1788696$.

What is the greatest product of four adjacent numbers in any direction (up, down, left, right, or diagonally) in the 20×20 grid?

```
A=matrix(20,20,[ 8, 2,22,97,38,15, 0,40, 0,75, 4, 5,
7,78,52,12,50,77,91, 8,
               49,49,99,40,17,81,18,57,60,87,17,40,98,43,69,48,
4,56,62, 0,
               81,49,31,73,55,79,14,29,93,71,40,67,53,88,30,
3,49,13,36,65,
               52,70,95,23, 4,60,11,42,69,24,68,56, 1,32,56,71,37,
2,36,91,

 22,31,16,71,51,67,63,89,41,92,36,54,22,40,40,28,66,33,13,80,
               24,47,32,60,99, 3,45,
2,44,75,33,53,78,36,84,20,35,17,12,50,

 32,98,81,28,64,23,67,10,26,38,40,67,59,54,70,66,18,38,64,70,
               67,26,20,68, 2,62,12,20,95,63,94,39,63,
8,40,91,66,49,94,21,
               24,55,58,
5,66,73,99,26,97,17,78,78,96,83,14,88,34,89,63,72,
               21,36,23, 9,75, 0,76,44,20,45,35,14,
0,61,33,97,34,31,33,95,
               78,17,53,28,22,75,31,67,15,94, 3,80, 4,62,16,14,
```

```
9,53,56,92,
               16,39, 5,42,96,35,31,47,55,58,88,24,
0,17,54,24,36,29,85,57,
               86,56, 0,48,35,71,89, 7,
5,44,44,37,44,60,21,58,51,54,17,58,
               19,80,81,68, 5,94,47,69,28,73,92,13,86,52,17,77,
4,89,55,40,
                4,52, 8,83,97,35,99,16,
7,97,57,32,16,26,26,79,33,27,98,66,
               88,36,68,87,57,62,20,72,
3,46,33,67,46,55,12,32,63,93,53,69,
                4,42,16,73,38,25,39,11,24,94,72,18,
8,46,29,32,40,62,76,36,
               20,69,36,41,72,30,23,88,34,62,99,69,82,67,59,85,74,
4,36,16,
               20,73,35,29,78,31,90,
1,74,31,49,71,48,86,81,16,23,57, 5,54,
                1,70,54,71,83,51,54,69,16,92,33,48,61,43,52,
1,89,19,67,48])
```